The Anatomy of Sentiment-driven Fluctuations*

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Abstract

We show that *sentiments* - self-fulfilling changes in beliefs that are orthogonal to fundamentals - can drive persistent aggregate fluctuations under rational expectations. Such fluctuations take place even in the absence of any exogenous aggregate shocks. In addition, sentiments also alter the volatility and persistence of aggregate outcomes in response to fundamental changes.

Keywords: sentiments, endogenous information, dynamic signal extraction
JEL classifications: E20, E32, F44

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1 Introduction

Can a change in sentiments induce persistent macroeconomic fluctuations? Even though this is a very attractive proposition and has captured the minds of economists at least since Keynes and Pigou, this idea has been very hard to formalize under rational expectations. We revisit this question in this paper.

We explore this question in the context of a general beauty contest model that has a unique Rational Expectations Equilibrium (REE) under full information. A key feature of the information structure is that agents receive noisy *endogenous* signals about the aggregate action in the economy. In particular, these noisy signals confound the information about current aggregate actions with payoff-relevant fundamentals and agents must filter out the average action in order to make inferences about their own appropriate action. Therefore, each agent's action depends on the aggregate action even if the primitives of the model do not feature any coordination motive. This induced strategic complementarity allows for persistent fluctuations driven by self-fulfilling changes in beliefs. We refer to these self-confirming changes in beliefs as *sentiments*, and aggregate fluctuations driven by these changes as *sentiment-driven fluctuations*. These sentiment-driven fluctuations are independent of changes in fundamentals such as technologies, preferences, or government policies. In fact, they can even exist in an economy without any change in these aggregate fundamentals and this fact is common knowledge.

It is crucial to note that our definition of sentiments is fundamentally *different* than the way the term is used in the fast growing theoretical and empirical literature which studies expectations-driven fluctuations.[1] This literature has largely modeled sentiments as an *exogenous* stochastic process which alter the agents' first-order beliefs or higher-order beliefs about fundamentals. As a result, these exogenous changes in *sentiments* can affect aggregate outcomes. In contrast, in the context of our model, changing sentiments are self-fulfilling changes in beliefs which arise *endogenously* in the sense that their evolution is disciplined by rational expectations.

Our notion of sentiments is most similar to Benhabib et al. (2015). It is important to point out that while Benhabib et al. (2015) provide an illustration of how sentiments can generate stochastic self-fulfilling rational expectations equilibria, they only consider a static environment and do not study whether sentiments can generate persistent fluctuations. In this paper, we overcome this restriction and show that sentiments can generate persistent aggregate fluctuations - a salient feature supported by the empirical findings (see Makridis (2017) and Lagerborg et al. (2018) among others).

Our first contribution is to provide general conditions under which a change in sentiments

[1]See for example Angeletos and La'O (2010, 2013), Lorenzoni (2009), Barsky and Sims (2012), Acharya (2013), Nimark (2014), Rondina and Walker (2014), Angeletos et al. (2014), Huo and Takayama (2015) among many others.

can have prolonged effects on aggregate outcomes, and when it can only have short-lived effects. Our analysis shows that if agents observe both (i) the history of realizations of past aggregate actions with a k-period lag and (ii) the history of realizations of past aggregate fundamentals with a k-period lag, then sentiments can drive fluctuations in such an economy with at most an MA($k-1$) process. The characterization of these conditions does not depend on the private information agents might possess. Particularly, if $k = 1$, the sentiment equilibrium can only be i.i.d like, where consistency between the perceived and actual laws of motion that REE demands rules out any possibility of persistent effects of sentiments.

Thus, a necessary condition for sentiments to have long-lasting fluctuations is a failure of one of the two conditions mentioned above. However, in our setting, the standard way of violating these conditions by imposing additional exogenous noises may not work. The reason is that our notion of sentiment equilibrium relies on the existence of endogenous signals, where a finite-state analytical solution typically does not exist when additional shocks are added. The possibility of multiple sentiment equilibria and the existence of other equilibria without sentiments further complicate the issue.[2]

Our second contribution is to provide a convenient modification that breaks the two conditions above and still allows for an analytical solution. We assume that agents do not observe past aggregate actions or fundamentals perfectly, but only a weighted average of them. This kind of signal can induce persistent forecast errors without adding any additional noises. We show that even without any fundamental shocks in the economy, the sentiment shock alone can generate not only aggregate fluctuations, but also persistent ones. Note that when the sentiment shock is muted, there is a unique equilibrium where the aggregate action is constant through time.

With aggregate fundamental shocks, the sentiment shock alters how an economy responds to fundamental shocks. Notably, the response to the fundamental shock features additional persistence, and is possibly hump-shaped. The sentiment shock therefore can be interpreted as an aggregate noise shock, and agents need to infer fundamentals via Bayesian learning. In an extreme case, even when the aggregate fundamental is i.i.d, the resulting fluctuations can be persistent. Thus, sentiments might also be able to act as amplification mechanisms with regard to fundamental shocks. We should emphasize again that the laws of motions for the sentiment-driven fluctuations are shaped by the equilibrium endogenously.

Theoretically, we extend the analysis of sentiments to a dynamic setting, and prove the existence of persistent sentiments by construction. This can be viewed as an alternative micro-foundation for fluctuations that are driven by non-fundamentals. For the applied work, we show the dynamics of sentiment equilibria resemble those with exogenous shocks to expectations in numerical examples. In the future, it can also be brought to a quantitative model.

[2] Numerical results are insufficient to prove the existence of sentiment equilibrium.

Related Literature Our notion of sentiments is mostly closely related to the work by Benhabib et al. (2015) in which they restrict attention to i.i.d sentiments. Our paper overcomes this restriction and explores its dynamic implications. Another closely related paper is Chahrour and Gaballo (2016), where the sentiment is interpreted as the limit of a fundamental equilibrium where the variance of the fundamental shock goes to zero. Unlike their paper, we go beyond a static environment, and our sentiment shock does not hinge on the existence of aggregate fundamental shock in the first place.

In the literature of dispersed information, sentiments, confidence, or animal spirits are often modeled as exogenous shocks to agents' expectations. For example, common noises or measurement errors to the observation of aggregate fundamentals are are exogenous shocks to agents' first-order beliefs about the fundamental, such as in Angeletos and La'O (2010) and Barsky and Sims (2012) among many others. The sentiment shock in Angeletos and La'O (2013) instead alters agents' higher-order beliefs about the fundamental. Different from previous studies, our sentiments are not imposed by the information set in an ad-hoc way, but arises from and are disciplined by the rational expectations equilibrium endogenously. However, our endogenous sentiments can play a role similar to an exogenous common noise. The type of sentiments that the recent empirical literature (Benhabib and Spiegel, 2017; Lagerborg et al., 2018; Makridis, 2017) attempts to identify is also akin to our notion of sentiments, which are completely uncorrelated with fundamentals.

The sentiment equilibria that we obtain are also related to correlated equilibria of Aumann (1974), as further developed by Maskin and Tirole (1987).[3,4] In Maskin and Tirole (1987), there exists a unique fundamental equilibrium and correlated equilibria exist only if there are Giffen goods. In our model, all goods can be normal and demand functions downward sloping, as in Benhabib et al. (2013, 2015). In a linear Gaussian economy, Bergemann and Morris (2013) and Chahrour and Ulbricht (2017) characterize the set of correlated equilibria and construct the corresponding information process (without sentiment shocks) that supports a particular allocation in the set. Our exercise instead starts from a particular information structure, and explore the set of equilibria that can be supported by the given primitives.

Sentiment-driven fluctuations, in our paper, took the form of self-confirming beliefs about aggregate outcomes. Thus, one could interpret the sentiment equilibria as sunspots. However, it is important to realize that the continuum of sentiment equilibria that we characterized are not simple sunspot randomizations over multiple fundamental equilibria as in many macroeconomic models. On the other hand, there exists a significant literature showing that sunspot equilibria

[3] See also Peck and Shell (1991), example 5.7.

[4] Aumann et al. (1988) provide an excellent overview of the relation between correlated and sunspot equilibria under asymmetric information with a set of examples in market games that in the limit converge to a competitive equilibrium, and also illustrate that under asymmetric information there can be correlated equilibria even though the fundamental equilibrium is unique.

can occur in models where the fundamental equilibrium is unique. The seminal paper of Cass and Shell (1983) demonstrates this in a two period model with a unique fundamental equilibrium by introducing securities traded in the first period, with returns that are sunspot contingent and can induce wealth effects. Peck and Shell (1991) obtain a similar result by postulating imperfect competition and non-Walrasian trades in the post-sunspot market that also gives rise to wealth effects.[5] By contrast Mas-Colell (1992) and Gottardi and Kajii (1999) explicitly rule out securities with payoffs contingent on sunspot realizations, but trading is possible due to heterogeneous endowments and preferences in the first period. Thus according to Gottardi and Kajii (1999) what accounts for the existence of sunspot equilibria is "potential multiplicity" in future spot markets.[6] It is clear that these are not the forces generating the multiple sunspot equilibria in our economy, as agents do not trade assets and do not make any inter-temporal decisions. Instead, the multiple equilibria in our model arise due to signal extraction problems in a setting with endogenous information sources.

On the technical side, the way we relax the conditions in Proposition 1 to generate persistent sentiments is similar to Rondina and Walker (2014). Both their paper and ours require some parts of the information process to be non-invertible to prevent signals being fully revealing. The major difference is that because of sentiment shock, the impulse response in our economy does not display oscillations which is a typical pattern in their paper. The minor difference is that we impose the non-invertible component in the observation equation, while they assume the fundamental itself is non-invertible.

2 Environment and Equilibrium Concept

We consider a standard beauty-contest game such as in Morris and Shin (2002). Our economy consists of a continuum of agents indexed by $i \in [0, 1]$. Agent i wants to choose an action $a_{i,t}$ every period which depends on their idiosyncratic fundamental shock $z_{i,t}$, an aggregate fundamental shock θ_t and the economy wide aggregate action a_t. Assume that the optimal action by agent i is given by:

$$a_{i,t} = \alpha \mathbb{E}[z_{i,t} \mid \mathcal{I}_{i,t}] + \varphi \mathbb{E}[\theta_t \mid \mathcal{I}_{i,t}] + \gamma \mathbb{E}[a_t \mid \mathcal{I}_{i,t}], \tag{1}$$

where

$$a_t = \int a_{i,t} \tag{2}$$

[5]See also Spear (1989) for an overlapping generations model with two islands where prices in one island act as sunspots for the other.

[6]Mas-Colell (1992) and Gottardi and Kajii (1999) give examples of such economies characterized by endowments, preferences and security payoffs. Gottardi and Kajii (1999) also provide a systematic method to generically construct such economies with sunspot equilibria.

is defined as the aggregate action and $\mathcal{I}_{i,t}$ denotes the information set of agent i at date t. α and φ can take any value on the real line but we impose that the $\gamma < 1$. This assumption ensures that there is a unique full-information fundamental equilibrium.[7] The processes for idiosyncratic and aggregate fundamental are given by:

$$z_{i,t} = \mathbf{h}(L)\mathbf{u}_{i,t} = \sum_{k=0}^{\infty} \mathbf{h}_k \mathbf{u}_{i,t-k} \tag{3}$$

$$\theta_t = \mathbf{g}(L)\mathbf{v}_t = \sum_{k=0}^{\infty} \mathbf{g}_k \mathbf{v}_{t-k}, \tag{4}$$

where $\mathbf{u}_{i,t}$ and $\mathbf{v}_t$ are sequences of Gaussian white noise innovations to the idiosyncratic and aggregate fundamental respectively.[8] $\mathbf{u}_{i,t}$ is a vector of idiosyncratic shocks to agents' fundamental and satisfies an adding-up constraint $\int_i \mathbf{u}_{i,t} = \mathbf{0}$ at each date t. In contrast, $\mathbf{v}_t$ is common across all agents. Furthermore, we assume that $\mathbf{h}(L)$ and $\mathbf{g}(L)$ are potentially infinite-order one-sided polynomials in positive powers of the lag operator L.[9] We do not impose any restrictions on $\mathbf{h}(L)$ and $\mathbf{g}(L)$ except square-summability which implies that $z_{i,t}$ and θ_t are linear stationary processes. Also, note that for the rest of the paper, bold-face letters indicate vectors and matrices while non bold variables indicate scalars.

Information Set of Agents We impose very little structure on the information sets that each agent possesses. We allow for cases in which agents observe noisy signals about fundamentals. Agents in the model have access to both *exogenous* and *endogenous* sources of information. Exogenous sources of information are those that are not affected by interactions among agents. These are modeled as a set of exogenous signals $\mathbf{y}_{i,t}$ which take the form:

$$\mathbf{y}_{i,t} = \mathbf{P}(L)\boldsymbol{\nu}_t + \mathbf{Q}(L)\boldsymbol{\zeta}_{i,t} \tag{5}$$

where $\boldsymbol{\nu}_t = \begin{bmatrix} \mathbf{v}_t & \boldsymbol{\eta}_t \end{bmatrix}'$ and $\boldsymbol{\zeta}_{i,t} = \begin{bmatrix} \mathbf{u}_{i,t} & \boldsymbol{\varsigma}_{i,t} \end{bmatrix}'$. $\boldsymbol{\eta}_t$ represents the vector of noise which is common across agents. In the literature, $\boldsymbol{\eta}_t$ is often interpreted as noise shocks, animal spirits or confidence shocks.[10] Thus, the vector $\boldsymbol{\nu}_t$ contains both innovations to fundamentals $\mathbf{v}_t$ and also the *noise* shocks $\boldsymbol{\eta}_t$. In a similar fashion $\boldsymbol{\varsigma}_{i,t}$ denotes the vector of idiosyncratic noise which may confound an agent's ability to observe fundamentals. The distinction between $\boldsymbol{\eta}_t$ and $\boldsymbol{\varsigma}_{i,t}$ is that while $\boldsymbol{\eta}_t$ is common across all agents, $\boldsymbol{\varsigma}_{i,t}$ varies by agent. We collect both idiosyncratic

[7] γ is a measure of the strength of strategic complementarities. If $\gamma \geq 1$, this complementarity is strong enough to generate multiple equilibria. See for example Cooper and John (1988). Since we restrict $\gamma < 1$, our results do not depend on the strength of the strategic complementarity.

[8] Even though the idiosyncratic and aggregate fundamentals are univariate stochastic processes, we allow them to be driven by a vector of innovations.

[9] As is convention we define the lag operator L as $Lx_t := x_{t-1}$, $L^{-1}x_t := x_{t+1}$ and $L^n x_t = x_{t-n}$.

[10] See for example Lorenzoni (2009), Angeletos and La'O (2013), among many others.

fundamentals $\mathbf{u}_{i,t}$ and idiosyncratic noise $\boldsymbol{\varsigma}_{i,t}$ into the vector $\boldsymbol{\zeta}_{i,t}$. $\mathbf{P}(L)$ and $\mathbf{Q}(L)$ can be any square summable, one sided polynomials in the lag operator L.[11] This structure is very general and encompasses commonly used assumptions in models with information frictions. For example, consider a situation in which each agent observes a public and a private signal about the aggregate fundamental,

$$
\begin{aligned}
y^1_{i,t} &= \upsilon_t + \eta_t, \\
y^2_{i,t} &= \upsilon_t + \varsigma_{i,t},
\end{aligned}
$$

which is the similar to the specification in Morris and Shin (2002). In terms of equation (5), this information structure can be represented as:

$$
\mathbf{y}_{i,t} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{P}(L)} \begin{bmatrix} \upsilon_t \\ \eta_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{Q}(L)} \varsigma_{i,t}
$$

In contrast to *exogenous* sources of information, *endogenous* sources are affected by interactions among agents. In other words, the informativeness of such signals is determined in equilibrium. We model such sources of information as the set of signals $\mathbf{x}_{i,t}$:

$$
\mathbf{x}_{i,t} \quad = \mathbf{A}(L)a_t + \mathbf{B}(L)\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}, \tag{6}
$$

The key distinction between $\mathbf{x}_{i,t}$ and $\mathbf{y}_{i,t}$ is that while $\mathbf{y}_{i,t}$ provides information about objects which aren't determined as part of equilibrium, $\mathbf{x}_{i,t}$ provides an agent with information about objects which are shaped by equilibrium. The availability of such information to agents in an economy is not hard to motivate. For example, firms, in finalizing their production decisions use information about expected *aggregate demand*, which is readily available from surveys of consumer expectations. Alternatively, one could think of such signals as market research by each firm regarding the demand for its own product.

The set of signals $\mathbf{x}_{i,t}$ are linear combinations of current and past innovations and aggregate action. As before, the only restriction we impose is that $\mathbf{A}(L), \mathbf{B}(L)$ and $\mathbf{C}(L)$ be square-summable and one sided polynomials in the lag operator L. If $\mathbf{A}(L) \neq \mathbf{0}$, then agents observe signals which provide information directly about equilibrium actions and not just about changes in exogenous fundamentals. The amount of information $\mathbf{x}_{i,t}$ provides to the agent depends on the equilibrium. To see this clearly, consider the case in which $\mathbf{B}(L) = \mathbf{C}(L) = 0$ and $\mathbf{A}(L) = 1$, i.e. $x_{i,t} = a_t$. Suppose that in equilibrium, a_t responds one-for-one to changes in the aggregate

[11]In other words, the signals can only depend on past and current changes (not future) in the fundamental shocks.

fundamental θ_t. Then, observing $x_{i,t}$ provides agent i enough information to infer the realization of θ_t perfectly. In contrast, if in equilibrium a_t does not respond to changes in the aggregate fundamental θ_t, then observing $x_{i,t}$ does not provide the agent with any information about the realization of θ_t. Thus, the informativeness of signals $\mathbf{x}_{i,t}$ is determined as part of equilibrium rather than being exogenously specified.[12]

This specification of endogenous information is general enough to encompass assumptions that are made commonly in the information frictions literature. For example, suppose agents observe a public and a private signal about current and past aggregate action a_t and a_{t-1}:

$$\begin{aligned} x_{i,t}^1 &= a_t + \eta_t, \\ x_{i,t}^2 &= a_{t-1} + u_{i,t}. \end{aligned}$$

These signals can be written compactly in terms of equation (6) as:

$$\mathbf{x}_{i,t} = \underbrace{\begin{bmatrix} 1 \\ L \end{bmatrix}}_{\mathbf{A}(L)} a_t + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{B}(L)} \eta_t + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{C}(L)} u_{i,t}$$

It is important to notice that if $\mathbf{A}(L) = 0$, equation 6 encompasses the noisy public signal in equation (5). In principle, the vector of signals $\mathbf{x}_{i,t}$ can contain both endogenous and exogenous signals. Thus, even though it is not necessary, we define equation (5) separately from (6) because of the notational convenience. In summary, we can express the information set of any agent i at date t as:[13]

$$\mathcal{I}_{i,t} = \mathbb{V}\left(\mathbf{y}_i^t\right) \vee \mathbb{V}\left(\mathbf{x}_i^t\right) \vee \mathbb{M} \tag{7}$$

where $\mathbb{V}\left(\mathbf{y}_i^t\right)$ denotes the smallest sub-space spanned by (at date t) by the past and current realizations of exogenous information y_i^t. $\mathbb{V}\left(\mathbf{x}_i^t\right)$ is defined analogously but for endogenous sources of information $\mathbf{x}_i^t$. Finally, since we are going to concentrate only on rational expectations equilibria, all agents have knowledge of the so-called *cross-equation restrictions* imposed by a rational expectations equilibrium. This is denoted by $\mathbb{M}$ and simply means that the agent knows that the dynamics of the economy are determined by equations (1) - (4).[14]

[12] Notice that setting $\mathbf{A}(L) = 0$ shuts off this property of endogenous informativeness of the signal.

[13] $\mathbf{X} \vee \mathbf{Y}$ denotes the smallest closed subspace which contains the subspaces $\mathbf{X}$ and $\mathbf{Y}$.

[14] In the paper we assume that agents cannot observe the current fundamentals perfectly when making their decisions. Although this is not key, we will generally assume that the information available to agents will not be sufficient to infer the aggregate and idiosyncratic fundamentals perfectly.

Equilibria In this paper, we focus on linear rational expectations equilibrium (REE). We classify these REE into two broad classes which are labeled as *fundamental* equilibrium and *sentiment* equilibrium. By fundamental equilibrium, we refer to those equilibria in which the aggregate action a_t is driven solely by exogenous aggregate shocks and is formalized in the definition below.

Definition 1 (Fundamental Equilibrium)**.** *In any fundamental equilibrium, the aggregate action is driven purely by changes in the aggregate fundamental innovations $\mathbf{v}_t$ and common noise $\boldsymbol{\eta}_t$ in the exogenous information:*

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t \tag{8}$$

where $\boldsymbol{\psi}(L)$ is a vector of square-summable rational polynomial in positive powers of the lag operator L. Furthermore, a_t is consistent with the agents' optimal choice given the information set $\mathcal{I}_{i,t}$ in (7)

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t = \int \Big\{\alpha\mathbb{E}[z_{i,t} \mid \mathcal{I}_{i,t}] + \varphi\mathbb{E}[\theta_t \mid \mathcal{I}_{i,t}] + \gamma\mathbb{E}[a_t \mid \mathcal{I}_{i,t}]\Big\} di \tag{9}$$

In a fundamental equilibrium, aggregate fluctuations are driven solely by changes in exogenous fundamentals of the economy.[15] For example, these exogenous shocks can be aggregate TFP or preference shocks. Furthermore, we allow fundamental equilibria to include those in which agents may not directly observe the fundamentals θ_t. In such a setting, aggregate noise in signals can also result in aggregate fluctuations. Thus, this class of equilibria encompasses the standard full-information equilibrium as well as those in economies with information frictions. In the latter, the definition is general enough to include both equilibria with exogenous information and endogenous information.

Definition 2 (Sentiment Equilibria)**.** *Consider any payoff irrelevant white noise process $\{\epsilon_t\}$ where $\{\epsilon_t\} \perp \{\boldsymbol{\nu}_t, \boldsymbol{\zeta}_{i,t}\}$. A sentiments equilibrium is one in which the aggregate action is driven by changes in fundamental innovations $\mathbf{v}_t$, exogenous noise $\boldsymbol{\eta}_t$ and also by changes in payoff irrelevant sentiments ϵ_t:*

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t \tag{10}$$

where $\epsilon_t \sim N(0,1)$. $\boldsymbol{\psi}(L)$ and $\phi(L)$ are square-summable rational polynomials in positive powers of the lag operator L and additionally, $\phi(L)$ has no roots inside the unit circle.[16] Moreover, a_t

[15] All fundamental equilibria lie in the Hilbert space $\mathcal{H}(\mathbf{v}, \boldsymbol{\eta})$ (the space spanned by square-summable linear combinations of $\mathbf{v}^t$ and $\boldsymbol{\eta}^t$).

[16] The assumption that $\phi(L)$ has no roots inside the unit circle is the same as requiring that $\phi(L)$ is invertible only in positive powers of L. Proposition 4 in Appendix A shows that this is without loss of generality - if we consider an equilibrium in which $a_t = \boldsymbol{\psi}(L)\nu_t + \phi(L)\epsilon_t$ in which $\phi(L)$ is not invertible, then we can always construct another observationally equivalent equilibrium in which $a_t = \psi(L)\boldsymbol{\nu}_t + \widetilde{\phi}(L)\epsilon_t$ where $\widetilde{\phi}(L)$ is invertible and satisfies $\phi(L)\phi(L^{-1}) = \widetilde{\phi}(L)\widetilde{\phi}(L^{-1})$.

is consistent with the agents' optimal choice

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t = \int \Big\{\alpha\mathbb{E}[z_{i,t} \mid \mathcal{I}_{i,t}] + \varphi\mathbb{E}[\theta_t \mid \mathcal{I}_{i,t}] + \gamma\mathbb{E}[a_t \mid \mathcal{I}_{i,t}]\Big\}di \tag{11}$$

The key difference between the two classes is that in addition to fluctuations driven by forces in a fundamental equilibrium, the sentiments equilibria allows for aggregate fluctuations to also arise due to changes in a payoff irrelevant factor ϵ_t.[17] Notice that ϵ_t is completely unrelated to changes in fundamentals $\boldsymbol{\nu}_t$ and exogenous noise $\boldsymbol{\eta}_t$. Strictly speaking, sentiment equilibria are correlated equilibria.

In order to explain what sentiments are, it is useful to explain what they are not. As explained above, it may be the case that agents only observe aggregate fundamentals $\mathbf{v}_t$ with measurement error or noise $\boldsymbol{\eta}_t$ and this noise itself may drive aggregate fluctuations. According to our definition, this is *not* an example of sentiments-driven fluctuations. Sentiments ϵ_t, as we have defined them, must be orthogonal to the vector $\boldsymbol{\nu}_t$, which includes $\boldsymbol{\eta}_t$ - they are not part of an agent's exogenous sources of information $\mathbf{y}_{i,t}$. As another example, consider a game which features multiple equilibria in which all agents observe a public randomization device, like a sunspot. Such an environment might permit an equilibrium in which agents use this device to coordinate their actions, and thus the aggregate outcome responds to the coordination device. Again according to our definition this is not a sentiments-driven equilibrium since our environment features a unique full-information equilibrium and in addition our model does not necessarily feature a coordination motive. Furthermore, unlike the realization of such a public coordination device, ϵ_t is not part of the exogenous information set: therefore it is not observed directly; nor is it an exogenously given feature of the environment.

Given that exogenous sources of information $\mathbf{y}_{i,t}$ provide no information about ϵ_t, it follows that agents can only get information about ϵ_t through the endogenous sources of information $\mathbf{x}_{i,t}$. Moreover, even within $\mathbf{x}_{i,t}$, the only way ϵ_t affects an agent's information set is through the aggregate action a_t. This is in contrast to $\boldsymbol{\eta}_t$ which can appear independently of a_t in $\mathbf{x}_{i,t}$. Thus, unlike $\boldsymbol{\eta}_t$, ϵ_t is an *endogenous* source of aggregate fluctuations. The next section explains why such sentiment-driven fluctuations are possible even under rational expectations. In particular, our main focus in this paper is to study whether sentiments can drive persistent fluctuations.

3 Forces at Play

Now that we have defined the environment, we begin presenting a simple example to uncover which ingredients are essential in generating sentiment-driven fluctuations and which ingredients

[17] All sentiment equilibria lie in the Hilbert space $\mathcal{H}(\mathbf{v}, \boldsymbol{\eta}, \epsilon)$ (the space spanned by square-summable linear combinations of $\mathbf{v}^t, \boldsymbol{\eta}^t$ and ϵ^t).

are not. Before exploring how and when sentiments can generate persistent fluctuations, we first show how sentiment-driven fluctuations can arise in the first place. For this purpose, we can ignore the time dimension of the environment we just specified. Recall that the best response of an agent i was given by (1) which is reproduced below for convenience

$$a_i = \alpha \mathbb{E}[z_i \mid \mathcal{I}_i] + \varphi \mathbb{E}[\theta \mid \mathcal{I}_i] + \gamma \mathbb{E}[a \mid \mathcal{I}_i]$$

As mentioned earlier, the assumption that $\gamma < 1$ ensures that the strength of strategic complementarity in decisions is not strong enough to generate multiple equilibria as in Cooper and John (1988). For this example, further assume that the aggregate fundamental θ is constant at $\theta = 0$ all the time and that this fact is common knowledge. Further, assume that the idiosyncratic fundamental z_i is i.i.d. across agents. In this case, the best response function simplifies to:

$$a_i = \alpha \mathbb{E}[z_i \mid \mathcal{I}_i] + \gamma \mathbb{E}[a \mid \mathcal{I}_i] \tag{B.1}$$

where $a = \int a_i di$ denotes the aggregate or average action. We assume that each agent i observes a noisy signal endogenous signal:

$$x_i = a + z_i$$

The signal x_i provides agent i with some information about the aggregate action a but is contaminated by the idiosyncratic fundamental z_i. The important thing about the signal x_i is not the exact form it takes,[18] but rather the fact that it provides agents with information about the endogenous variable a_t. In this setting, it is straightforward to see that the optimal action of agent i must take the form:

$$a_i = \varpi x_i = \varpi a + \varpi z_i \tag{12}$$

where the second equality follows from the definition of x_i. Importantly, the constant ϖ is determined as part of equilibrium. Aggregating the decisions of all agents:

$$a = \int a_i di = \varpi a + \varpi \int z_i di = \varpi a \tag{13}$$

Equation (13) implies that in equilibrium, it must be the case that $a(1 - \varpi) = 0$. Notice that if $\varpi \neq 1$, then the only equilibrium is one in which $a = 0$ for all t. This is the unique *fundamental equilibrium*. But if $\varpi = 1$, then the equation above is satisfied for any level of a!

Of course ϖ is determined as part of equilibrium and thus, it remains to show whether in equilibrium whether ϖ equals 1 or not . In order to understand what influences the equilibrium

[18]For example, it is straightforward to incorporate additional exogenous noise into this signal.

magnitude of ϖ, we conjecture that in equilibrium, the aggregate action can be described by:

$$a = \phi\epsilon \tag{14}$$

with the understanding that ϕ is also determined as part of equilibrium. We refer to ϵ as the "*sentiment*". $\epsilon \sim N(0,1)$ is independent of idiosyncratic and aggregate fundamentals. Plugging (14) into the expression for x_i yields:

$$x_i = z_i + \phi\epsilon \tag{15}$$

The signal x_i provides agent i information about their idiosyncratic fundamental z_i but is potentially contaminated by the sentiment ϵ. Importantly, the precision of this signal depends on ϕ - with $\phi = 0$, the signal is fully informative about z_i; if $\phi \neq 0$, x_i is a noisy signal of z_i.

Verifying the Fundamental Equilibrium The *fundamental equilibrium* is one in which $\phi = 0$, i.e. the aggregate outcome is $a = 0$ and is unaffected by sentiments ϵ. With $\phi = 0$, the signal x_i in (15) perfectly informs agent i about the actual realization of her idiosyncratic fundamental, i.e, $\mathbb{E}[z_i \mid x_i] = z_i$. Consequently, using (B.1) agent i's optimal action is given by:[19]

$$a_i = \alpha\mathbb{E}[z_i \mid \mathcal{I}_i] = \alpha x_i \qquad \Rightarrow \qquad a = \int a_i di = \alpha \int z_i di = 0$$

Also, from (12) $\varpi = \alpha \neq 1$, confirming that $\phi = 0$ is indeed an equilibrium.

Can $\varpi = 1$ in equilibrium? We just saw that with $\phi = 0$, $\varpi \neq 1$. Thus, if there exist equilibria with $\varpi = 1$, it must be with $\phi \neq 0$. Then, unlike in the fundamental equilibrium, (15) does not have infinite precision,[20] and the signal does not allow an agent to perfectly infer the realization of z_i:

$$\mathbb{E}\left[z_i \mid x_i\right] = \frac{\sigma_z^2}{\sigma_z^2 + \phi^2} x_i \qquad \text{and} \qquad \mathbb{E}\left[a \mid x_i\right] = \frac{\phi^2}{\sigma_z^2 + \phi^2} x_i$$

Plugging these expressions into (B.1) and comparing with (12), it follows that:

$$\varpi = \alpha \frac{\sigma_z^2}{\sigma_z^2 + \phi^2} + \gamma \frac{\phi^2}{\sigma_z^2 + \phi^2} \tag{16}$$

[19] Agents know that $a = 0$ in this equilibrium and so $\mathbb{E}[a \mid \mathcal{I}_i] = 0$.

[20] In fact the signal-to-noise ratio, σ_z^2/ϕ^2 is decreasing in ϕ.

Then, for $\varpi = 1$, it must be the case that:

$$|\phi| = \sigma_z \sqrt{\frac{\alpha - 1}{1 - \gamma}} \tag{17}$$

Thus, in addition to the fundamental equilibrium, the endogenous signal also supports an additional *sentiment* equilibrium in which sentiments can affect aggregate outcomes a even though it is common knowledge that the aggregate fundamental $\theta = 0$![21]

Sentiment-driven Fluctuations and Correlated Forecast Errors Agents observe correlated signals in the sentiment equilibrium ($\phi \neq 0$): agent i and $j \neq i$ observe correlated signals since both x_i and x_j depend on a. In equilibrium, $cov(x_i, x_j) = \phi^2$ for $i \neq j$. Consequently, agents form correlated forecast errors about the aggregate action and can be written as:[22]

$$\mathbb{E}\Big[(\mathbb{E}\left[a \mid \tilde{x}_i\right] - a)(\mathbb{E}\left[a \mid \tilde{x}_j\right] - a)\Big] = \left[\frac{\sigma_z^2 \phi}{\sigma_z^2 + \phi^2}\right]^2 > 0 \text{ for } i \neq j$$

Since agents make correlated forecasts (and forecast errors) of a and z_i, their choices of a_i are also correlated in the sentiments equilibrium. Thus, even upon aggregating across all agents, this correlated component does not vanish. In contrast, in the fundamental equilibrium where $\phi = 0$, agents receive effectively uncorrelated signals, make uncorrelated forecasts (and forecast errors) and as a result when averaged across all agents, the forecast errors vanish.

Importantly the magnitude of $\gamma < 1$ (strength of the strategic complementarity) does not play any role in generating the sentiment equilibrium and can exist even if $\gamma = 0$ (no complementarity) or $\gamma < 0$ (strategic substitutability). Endogenous information induces complementarities even when the primitive economy may not feature any. To see this, notice that the equilibrium covariance between $a_{i,t}$ and a_t can be expressed as: $\mathbb{E}[a_i, a] = \sigma_z^2 \left(\alpha - 1\right) / \left(1 - \gamma\right)$ which is positive in a sentiments equilibrium even if $\gamma = 0$.[23] In contrast, in the fundamental equilibrium, this covariance is 0.

Adding aggregate fundamentals to the mix The example above showed that sentiments can affect outcomes independently of aggregate fundamentals. More generally, not only can sentiments affect aggregate outcomes, they can also affect how the economy responds to changes in aggregate fundamentals as the following example shows. For this demonstration it is convenient

[21] Such an equilibrium exists as long as $\alpha > 1$.

[22] The forecast error made by agent i can be written as $\mathbb{E}_i a - a = \frac{\phi \sigma_z^2}{\sigma_z^2 + \phi^2} \epsilon + \frac{\phi^2}{\sigma_z^2 + \phi^2} z_i$

[23] This is true if $\alpha > 1$ which is also the condition for the sentiment equilibrium to exist.

to assume that $z_{i,t} = 0$ for all i, t. The best response function can be specialized to:

$$a_i = \varphi \mathbb{E}[\theta \mid \mathcal{I}_i] + \gamma \mathbb{E}[a \mid \mathcal{I}_i] \tag{B.2}$$

Further assume that the endogenous signal now takes the form:

$$x_i = a + \zeta_i \tag{18}$$

where ζ_i is private noise. Again, in this setting, agent i's optimal decision must take the form $a_i = \varpi x_i$. In equilibrium, now the aggregate outcome can respond to both aggregate fundamentals θ and sentiments ϵ. As before, we start by conjecturing that in equilibrium, the aggregate action can be written as:

$$a = \psi\theta + \phi\epsilon$$

where ψ and ϕ are determined as part of equilibrium. Plugging in the equilibrium a in to the expression for the signal x_i yields:

$$x_i = \psi\theta + \phi\epsilon + \zeta_i$$

The expression above makes it clear that the informativeness of x_i about θ depends on the equilibrium ϕ and ψ - the signal to noise ratio is given by $\frac{\psi^2\sigma_\theta^2}{\phi^2+\sigma_\zeta^2}$ which is increasing in ψ and decreasing in ϕ. Appendix B shows that there is a continuum of equilibria in which the pair (ψ, ϕ) satisfy:

$$\left(\psi - \frac{\varphi}{2(1-\gamma)}\right)^2 + \frac{\phi^2}{\sigma_\theta^2} = r^2$$

where r is a constant which depends on the parameters of the model and is defined in Appendix B. Noticeably, one of the equilibria is the fundamental equilibrium in which $\phi = 0$ and all the aggregate fluctuations are accounted for by aggregate fundamentals. However, all other equilibria are sentiment equilibria in which $\phi \neq 0$ and thus sentiments also cause aggregate fluctuations. Importantly, the expression above makes clear that in an equilibrium where $|\phi|$ is large, the effect of changes in aggregate fluctuations is smaller. In other words, if agents in the economy believe that sentiments are important drivers of fluctuations, then the economy's response to changes in aggregate fundamentals can be diminished!

4 Beyond Static Sentiment Equilibrium

While the simple examples above abstracted from the time dimension, they illustrated that sentiments could result in aggregate fluctuations contemporaneously even if aggregate fundamentals were unchanged. Moreover, the presence of sentiments could even alter the strength of the contemporaneous effect of changes in aggregate fundamentals on aggregate outcomes. One way to interpret the examples above is that we have effectively restricted the sentiments to only affect aggregate outcomes contemporaneously (in terms of the notation from Section 2, we have restricted $\phi(L) = \phi_0$).

Next, we do not impose this restriction and explore whether the same forces can generate persistent sentiment-driven aggregate fluctuations, i.e. we study equilibria in which $\phi(L) = \sum_{k=0}^{\infty} \phi_k L^k$ where at least one $\phi_k \neq 0$ for $k > 0$. In other words, we now explore whether current changes in sentiments ϵ_t can affect aggregate outcomes in the future. We begin by establishing some general properties of sentiment equilibria when moving to this dynamic setting. To proceed, it is useful to define two assumptions on the information set

Assumption 1. *The past aggregate action is observable with a k-periods lag, i.e,* $a^{t-k} \in \mathcal{I}_{i,t}$.

Assumption 2. *The past exogenous aggregate shock with a k-period lag can be perfectly inferred from exogenous information, i.e,* $\mathbb{V}(\mathbf{y}_i^t) \supseteq \mathbb{V}(\boldsymbol{\nu}^{t-k})$.

If Assumption 1 is satisfied, at any date t each agent i observes the aggregate outcomes up till date $t-k$, i.e. each agent knows the sequence a^{t-k}. Similarly, Assumption 2 ensures that at any date t, each agent knows the realization of the innovation to the aggregate fundamentals ($\boldsymbol{\nu}$) up till date $t-k$, i.e at date t each agent knows the sequence $\boldsymbol{\nu}^{t-k}$.[24] With these assumptions in place, we are ready to characterize equilibria in which sentiments can have persistent effects.

Proposition 1. *If Assumptions 1 and 2 are satisfied, then in any sentiment equilibria* $\phi(L) = \sum_{\tau=0}^{\infty} \phi_\tau L^\tau$*, it must be the case that* $\phi_\tau = 0$ *for all* $\tau > k$*. In other words, a change in sentiments* ϵ_t *at date* t *cannot affect outcomes after date* $t + k$.

Proof. See Appendix C.1 for the proof. □

Proposition 1 states that if agents observe past aggregate action and fundamental perfectly with a finite lag k, the effects of a change in sentiments dies out after a finite number of periods. In particular, dynamics of the aggregate outcome driven by changes in sentiments can be described by a moving average process where the maximum lag length is k.

A few remarks are in order at this point. First, the statement of Proposition 1 holds under very general conditions as we do not impose any restrictions on the number of shocks, the number of signals or private information that each agent might possess.[25] Second, even though Assumptions 1-2 imply that agents can observe past aggregate actions and fundamentals, there

[24] Notice that for Assumption 2 to be satisfied, it is not necessary for agents to observe $\boldsymbol{\nu}_{t-k}$ directly. For example, if the aggregate fundamental follows an AR(1) process, $\theta_t = \rho\theta_{t-1} + v_t$, observing past fundamentals $\{\theta^{t-1}\}$ allows agents to infer past shocks $\{v^{t-1}\}$ perfectly.

[25] We also do not impose restrictions on whether the signal process is invertible or non-invertible.

is no supposition that they observe idiosyncratic fundamentals perfectly. In fact, agents can still have persistent forecast errors about their individual fundamentals. However, Proposition 1 makes clear that these forecast errors about the idiosyncratic fundamental *cannot* translate into persistent aggregate fluctuations. existence.

Corollary 1. *Particularly, if $k = 1$, the sentiment equilibria $\phi(L)$ can only be a constant, i.e., $\phi(L) = \omega$.*

A direct corollary of Proposition 1 is that if at any date t, each agent observes the realization of the aggregate fundamental θ_{t-1} and a_{t-1}, then the *only* sentiment equilibrium if one in which changes in sentiments at date t can only affect aggregate outcomes contemporaneously, i.e., $\partial a_{t+s}/\partial \epsilon_t = 0$ for all $s > 0$. This result is independent of any private information that agents may possess or other signals that they might observe. Corollary 1 also implies that the static examples in the previous section can equally be interpreted in terms of a dynamic environment in which each agent observes (or can infer) θ_{t-1} and a_{t-1} at date t. The upshot of Proposition 1 and in particular Corollary 1 is that in order for sentiment driven fluctuations to display persistence, Assumption 1 and/or Assumption 2 do not hold for $k = 1$, i.e. $a^{t-1} \notin \mathcal{I}_{i,t}$ and/or $\mathbb{V}(\mathbf{y}_i^t) \not\supseteq \mathbb{V}(\nu^{t-1})$. This powerful characterization provides a helpful insight to the large literature which studies sentiment-driven equilibria such as Benhabib et al. (2013, 2015) and Chahrour and Gaballo (2016) among others. While this literature has largely concentrated on studying i.i.d fluctuations driven by sentiments, Proposition 1 and Corollary 1 serve as a guide by uncovering the minimum ingredients required to construct and study equilibria in which sentiments can drive persistent fluctuations. The results above also provide additional insight about the large literature which studies models with information frictions. While this literature uses models which are very similar to the setting studied in this paper, their focus has largely been on fundamental equilibrium. In order to avoid the complexity of dealing with the problem referred to as *forecasting the forecasts of others*, Townsend (1983), researchers in this literature have commonly made the assumption that the realizations of aggregate fundamentals and aggregate outcomes in the past are common knowledge. Proposition 1 shows that these assumptions on the information set of agents rules out the possibility of persistent sentiment-driven fluctuations.

Finally, it is important to note that Proposition 1 is not about the existence of sentiment equilibrium; the statement of Proposition 1 is conditional on a sentiment equilibrium existing. This raises the question whether there exist any equilibria in which sentiments can drive persistent and predictable aggregate fluctuations even if we relax these assumptions. We show that this is in fact the case by presenting two examples of such equilibria when we relax Assumptions 1 and 2. These examples show that the set of such sentiment equilibria is not empty.

Example 1: a_{t-1} is not observed at date t. We start with the dynamic counterparts to the example studied in Section 3 where the aggregate fundamental was fixed at $\theta_t = 0$ for all time and this is common knowledge. The best-response of agent i at any date t can be written as:

$$a_{i,t} = \alpha \mathbb{E}[z_{i,t} \mid \mathcal{I}_{i,t}] + \gamma \mathbb{E}[a_t \mid \mathcal{I}_{i,t}]$$

which is identical to B.1 except that we have appended time-subscripts. In this setting Assumption 2 is trivially satisfied in the sense that at date t,all agents know the realization of the aggregate fundamental at $t-1$. It follows from Proposition 1 that if agents observed a_{t-1} at date t, then the only sentiment equilibrium takes the form:

$$a_t = \phi \epsilon_t$$

where ϕ is defined in equation (17). Now relax this assumption and assume that at date t, agents cannot observe the realization of a_{t-1} but can observe the aggregate outcome with two lags. This can be formalized as each agent i receiving two signals at each date:

$$x^1_{i,t} = a_t + z_{i,t} \qquad \text{and} \qquad x^2_{i,t} = a_{t-2}.$$

where the first signal is a private signal as in Section 3 except that we have appended time-subscripts. Applying the statement of Proposition 1, a sentiment equilibria in which $\phi(L)$ is a MA(1) can exist. In fact, Appendix D.1 shows that there exist a sentiment equilibrium in which the aggregate outcome at any date t is affected by contemporaneous changes in sentiments ϵ_t and by yesterday's changes in sentiments ϵ_{t-1}.

$$a_t = \phi_0 \epsilon_t + \phi_1 \epsilon_{t-1}.$$

The expressions describing ϕ_0 and ϕ_1 are contained in Appendix D.1. This example shows that relaxing the conditions in Proposition 1 does in fact allow for sentiment equilibria in which sentiments can drive persistent aggregate fluctuations although in a limited fashion.

Example 2: θ_{t-1} is not observed at date t For this example we consider the other example in Section 3 where aggregate fundamentals were stochastic but idiosyncratic fundamentals were fixed at $z_{i,t} =$ for alli, t. Recall that the best-response function at date t in this case can be written as:

$$a_{i,t} = \varphi \mathbb{E}_{it}[\theta_t] + \gamma \mathbb{E}_{it}[a_t]$$

which is identical to equation (B.2) except for the additional time-subscripts. It is clear from Proposition 1 that if agents observed θ_{t-1} at date t, sentiments can only affect contemporaneous aggregate outcomes. We relax this by assuming that agents only observe the realization of the aggregate fundamental with a lag of two periods. The signal structure observed by agent i in this case can be written as:

$$x^1_{i,t} = a_t + \zeta_{i,t}, \qquad x^2_{i,t} = a_{t-1} \qquad \text{and} \qquad y^1_{i,t} = \theta_{t-2}$$

where as in Section 3 $\zeta_{i,t}$ denotes noise in the private signal $x^1_{i,t}$. The signal $x^2_{i,t}$ implies that agents at date t perfectly observe the aggregate outcome at date $t-1$. While the first two signals are endogenous, the third is exogenous and implies that agents only observe the realization of the aggregate fundamental with a lag of two periods. In this setting, Appendix D.2 confirms that there exist a continuum of equilibria in which current and yesterday's changes in sentiments can affect aggregate outcomes alongside change in aggregate fundamentals. In fact, the evolution of the aggregate outcome a_t can be expressed as:

$$a_t = \phi_0 \epsilon_t + \phi_1 \epsilon_{t-1} + \psi_0 v_t + \psi_1 v_{t-1},$$

where v_t denotes the date t innovation to the aggregate fundamental θ. Appendix D.2 shows that in equilibrium, the coefficients satisfy: $\phi_0^2 + \psi_0^2 + \phi_1^2 + \psi^2 = \frac{\alpha-1+\varphi\psi_0}{1-\gamma}$, $\phi_0\phi_1 = -\psi_0\psi_1$ with $\left|\frac{\phi_1}{\phi_0}\right| < 1$.

5 Persistent Sentiment-Driven Fluctuations

While from a theoretical point of view, we have proved the existence of persistent sentiment, this is less than satisfying for applied macroeconomic work which has found that sentiments can have long-lasting effects demonstrated by slowly decaying impulse responses. When Assumption 1 and 2 are satisfied, the perfect observation of the aggregate variables is only delayed by a finite number of periods, and this type of truncation forces the forecast errors to jump discretely to zero after a finite horizon. Consequently, this type of truncation forces the impulse response of aggregate outcomes to a sentiment shock to die out abruptly after a finite number of periods.

A simple strategy to "smooth" out the decay of forecast errors is to assume that agents observe past realizations of aggregate outcomes and aggregate fundamentals with additional noise. The presence of such noise would prevent agents from perfectly inferring the exact realizations of aggregate productivity. However, such a strategy would lead to an environment in which agents observe fewer signals than shocks. While this in itself is not a problem, this does severely restrict analytical tractability in characterizing equilibrium since it involves dynamic signal extraction in a "non-square system" in a setting with endogenous signals.[26] Consequently,

[26]See Nimark (2014) and Huo and Takayama (2017) for more discussion.

the lack of analytical tractability makes it very difficult to establish the existence of sentiment equilibrium and to characterize its properties in these general settings in which the forecast errors would decay smoothly.

Next, we adopt a different strategy which allow us to relax assumptions 1 and 2 so as to construct equilibria in which the aggregate outcome displays a long lasting impulse response to sentiment shocks which does not die out abruptly after a finite number of periods. It is attractive to study such equilibria as they are easier to map to empirical studies. The following brief digression describes this approach.

A Brief Digression Suppose a variable y_t follows an invertible stationary process

$$y_t = \phi(L)e_t,$$

and a signal x_t is observed every period which takes the form:

$$x_t = (L - \lambda)y_t \tag{19}$$

for $\lambda \in (-1, 1)$. Notice that for small enough λ ($|\lambda| < 1$), the signal x_t puts very little weight on the current realization of y_t and thus the agent is unable to infer y_t perfectly by observing the sequence of signals x^t.[27] In fact, Appendix E shows that for $k \geq 0$, the forecast error about y_{t-k} at date t can be written as:

$$y_{t-k} - \mathbb{E}[y_{t-k}|x^t] = \lambda^k \phi(\lambda)(1 - \lambda^2) \sum_{s=0}^{\infty} \lambda^s e_{t-s}$$

In other words the forecast error decays gradually at a rate which is proportional to λ and only converges to zero asymptotically, i.e. the agent only learns about the actual realization of y_{t-k} asymptotically. Thus, a signal of the form (19) is a convenient way to model a smooth decay of the forecast error and avoids the undesirable discreteness with which forecasts error vanish as in the previous section. Compared to the strategy of adding additional observation noise, this alternative signal structure allows us to generate smoothly decaying forecast errors and also affords analytical tractability.[28] Armed with this new modeling device, we revisit the now

[27] A higher λ increases implies a larger weight on the current y_t and increases the informativeness of the signal about y_t. Actually, when λ is larger than 1, the signal reveals the underlying shock perfectly.

[28] This strategy shares some similarity to the confounding process in Rondina and Walker (2014). In Rondina and Walker (2014), the variable y_t itself is assumed to follow a non-invertible process. Instead, we introduce the non-invertible component in the signal which prevents agents from inferring the underlying shock e_t perfectly after finite time. To be clear, while the modeling strategy looks superficially similar to Rondina and Walker (2014), they *do not* study self-fulfilling sentiment fluctuations or how the presence of sentiments can affect the dynamic response of the economy to aggregate fundamentals, which we show next.

familiar examples from Section 4.

Example 1: a_{t-1} is not observable at date t (but now with the new signal) We start with the case where there is no aggregate fundamental shock in the economy. Recall that the best response in this case was given by (B.1) (reproduced here for convenience):

$$a_{i,t} = \alpha \mathbb{E}_{i,t}[z_{i,t}] + \gamma \mathbb{E}_{i,t}[a_t]$$

where $z_{i,t} \sim \mathcal{N}(0, \sigma^2)$. As in Example 1 in Section 4, agents observe two signals.

$$x_{i,t}^1 = a_t + z_{i,t} \qquad \text{and} \qquad x_{i,t}^2 = (L - \lambda)a_t$$

with $|\lambda| < 1$. The first signal is the same as before but now instead of observing a_{t-2}, each agent now observes the signal of the form we discussed above. As before, the unique fundamental equilibrium of this economy, in the absence of fluctuations in the aggregate fundamental is simply given by $a_t = 0$ for all t. Now we move to the sentiment equilibrium which is defined formally in the following proposition:

Proposition 2. *For $\alpha > 1$, there exists a unique sentiment equilibrium in which the dynamics of the aggregate outcome can be described by:*

$$a_t = \sigma(1 - \lambda^2)\sqrt{\frac{\alpha - 1}{1 - \gamma}} \frac{1}{1 - \lambda L} \epsilon_t \tag{20}$$

Proof. See Appendix C.2 for the proof. □

In contrast to Example 1 in Section 4, now the aggregate dynamics can be described by an AR(1) and thus, do not die out abruptly as in that section. Notice that the persistence of the sentiment equilibrium, λ is the same parameter as in the second signal. As was the case in Section 4, observing a_{t-k} at date t generated a sentiment equilibrium with MA(k) dynamics. Similarly, here, the dynamics inherit the properties of the signal that allows us to violate Assumption 1 for $k = 1$. This equilibrium construction has the potential to rationalize the empirical evidence on the persistence effects of sentiments on macroeconomic outcomes.[29]

[29] Also, notice that agents' forecast errors also die out gradually in this case rather than abruptly after a finite number of periods. Denote the sequence of the impulse responses of the average forecast error $a_t - \int \mathbb{E}_{it}[a_t]$ to a sentiment shock as $\{f_k\}_{k=0}^{\infty}$. Then, we have $f_k \propto (1 - \lambda)^2 \lambda^k$. If λ is close to zero, agents know little about current aggregate action, but they can quickly learn the past sentiment shocks as time goes. Therefore, f_0 is relatively large, but f_k dies out fast. If λ is close to one, agents know more about the current sentiment shock ϵ_t, but they learn little about the current shock ϵ_t in the future. This makes the forecast error more persistent. The aggregate action inherits the property of the forecast error, and the persistence of the aggregate action is tied with λ.

In the sentiment equilibrium, the volatility of the aggregate outcome is given by

$$\mathbb{V}(a_t) = (1-\lambda^2)\sigma^2\frac{\alpha-1}{1-\gamma}$$

This expression reveals the key property of the sentiment equilibrium - the persistence and volatility of the aggregate outcomes are tightly linked through equilibrium. The expression above shows that if the sentiment equilibrium displays a very persistent response to changes in sentiments (high λ), then the unconditional variance of the aggregate outcome is lower.[30]

Finally, it is important to understand that this response of aggregate outcomes to change in sentiments is fundamentally different from Angeletos and La'O (2013) where the aggregate fluctuation is driven by exogenous aggregate noise shock to higher-order beliefs. Instead, sentiments driven fluctuations in our model arise due to self-fulfilling beliefs and hence can be thought of as arising "*endogenously*" as part of equilibrium.

Learnability In this environment with one fundamental equilibrium $a_t = 0$ and one sentiment equilibrium, it is natural to ask which of these equilibria (if any) are stable under learning. In a similar spirit as in Benhabib et al. (2015), we show next that the sentiment equilibrium is a stable one under learning. Suppose agents' perceived law of motion of the aggregate action follows an AR(1) process $a_t = \frac{c}{1-\lambda L}\epsilon_t$, where $c = 0$ reduces to the fundamental equilibrium. The actual law of motion will follow

$$a_t = \frac{c\left(\sigma^2\alpha\left(1-\lambda^2\right)^2+\gamma c^2\right)}{\left(c^2+\sigma^2\left(1-\lambda^2\right)^2\right)}\frac{1}{1-\lambda L}\epsilon_t.$$

The mapping $\mathcal{T}(c) \equiv \frac{c\left(\sigma^2\alpha\left(1-\lambda^2\right)^2+\gamma c^2\right)}{\left(c^2+\sigma^2\left(1-\lambda^2\right)^2\right)}$ has two fixed points corresponding to the fundamental and the sentiment equilibrium respectively. It turns out that as long as the sentiment equilibrium exists,[31] then the fundamental equilibrium is unstable while the sentiment equilibrium is stable. By the *E-stability principle* (Evans and Honkapohja, 2012), this implies the sentiment equilibrium is stable under adaptive learning.

Example 2: θ_{t-1} is not observable at date t (but now with the new signal) In Example 2 in Section 4 (where aggregate fundamentals were not degenerate) we had relaxed Assumption 2 by assuming that agents only observe the realizations of the aggregate fundamental θ_{t-2} at

[30]Notice that if $\lambda = 0$, then the second signals is simply $x^2_{i,t} = a_{t-1}$ and Proposition 1 dictates that the only sentiment equilibrium can be one in which the aggregate outcome only responds to contemporaneous changes in sentiments.. The expression for $\mathbb{V}(a_t)$ shows that this case observes the lowest persistence but the the highest unconditional volatility of the aggregate outcome.

[31]Recall that this requires that $\alpha > 1$ in this example.

date t. We now change the signal structure - agents are assumed to observe three signals:

$$x^1_{i,t} = a_t + \zeta_{i,t}, \qquad x^2_{i,t} = a_{t-1} \qquad \text{and} \qquad y^1_{i,t} = (L-\lambda)\theta_t$$

Notice again that the first two signals are the same as before but we have changed the third signal. The third signal is an imperfect observation about the aggregate fundamental shock, which relaxes Assumption 2 in order to generate sentiment equilibrium which displays persistence.

We start by first characterizing the fundamental equilibrium. In the fundamental equilibrium, the aggregate action is by definition only driven by the fundamental shock, $a_t = \psi(L)v_t$. Agents observe two public signals about a_t, which implies that the fundamental shock v_t can be perfectly inferred. As a result, there is no information frictions in this economy, and the the unique fundamental equilibrium features an aggregate action which tracks the aggregate fundamental θ_t perfectly:

$$a_t = \theta_t = g(L)v_t.$$

However, the belief among agents that sentiments also affect aggregate outcomes changes things dramatically. In the presence of shocks to sentiment, agents are no longer able to use the public signals to infer the aggregate shocks. Instead, all agents have to solve a dynamic signal extraction problem. The proposition below presents the properties of sentiment equilibria:

Proposition 3. *There exits a continuum of sentiment equilibria $a_t = \phi(L)\epsilon_t + \psi(L)v_t$:*

$$\phi(L) = \kappa + \frac{\Phi(\kappa)L}{1-\lambda L}, \tag{21}$$

$$\psi(L) = \frac{\varphi}{1-\gamma}g(L) - \frac{\Psi(\kappa)}{1-\lambda L}, \tag{22}$$

indexed by $\kappa \in \mathbb{R}_+$ where κ satisfies:

$$\left(\varphi\left(1-\lambda^2\right)g(\lambda)\right)^2\kappa^2 > 4\left(\sigma^2+(1-\gamma)\kappa^2\right)\left(\lambda^2\sigma^2+(1-\gamma)\kappa^2\right), \tag{23}$$

and $\Phi(\kappa)$ and $\Psi(\kappa)$ are constants depend on κ.[32]

Proof. See Appendix C.3 for the proof. □

Proposition 3 shows that there are a continuum of sentiment equilibria in which fluctuations driven by shocks to sentiments can be described by an ARMA(1,1) process. Thus, much like in Example 1 above , changes in sentiments can generate persistent fluctuations. However, unlike in the previous example, the existence of a sentiment equilibrium itself depends on the properties

[32] The analytic form of $\Phi(\kappa)$ and $\Psi(\kappa)$ are quite involved and are shown in Appendix.

of the aggregate fundamental θ_t. To see this, note that if the aggregate fundamental shock is shut-off, i.e. $g(L) = 0$, then no κ can be supported, which is also implied by Proposition 1.

Moreover, the properties of sentiment-driven fluctuations are both affected by (and also affect) fluctuations driven by changes in fundamentals. Without shocks to sentiments, we simply have $\psi(L) = g(L)$, i.e. the aggregate action tracks the aggregate fundamental θ_t. However, in the presence of shocks to sentiments, agents are now forced to infer the the change in sentiments and the change in fundamentals from the observed signals. Since agents are no longer able to infer the true realization of θ_t by observing the signals, the presence of sentiments affects how the economy responds to fundamental shocks. Here sentiments behave as "endogenously" arising noise which prevents agents from perfectly inferring the aggregate fundamentals by observing the endogenous signals. This change in response of the economy to fundamental shocks is reflected in the additional AR(1) term $\frac{\Psi(\kappa)}{1-\lambda L}$. Particularly, this new component could generate additional persistence of to the dynamics of fluctuations driven by fundamental shocks, a salient feature of most macroeconomic variables. In an extreme case, even when the fundamental follows an i.i.d process, i.e. $g(L) = 1$, the response to the fundamental shock in the sentiment equilibria can still be persistent even though in the fundamental equilibrium, a change in aggregate fundamental would only affect aggregate outcomes contemporaneously.

To further appreciate how the presence of sentiments can alter the dynamic response of the economy, we present the following numerical example. We assume that the fundamental process follows an AR(1) process: $\theta_t = \rho\theta_{t-1} + v_t$.[33] We set $\rho = 0.9$, $\lambda = 0.7$, $\gamma = 0.1$, $\varphi = 1-\gamma$, and $\sigma = 0.2$. As was mentioned earlier there are a continuum of sentiment equilibria indexed by κ which are consistent with this parameterization. Denote the set of κ which satisfies (23) as $[\underline{\kappa}, \overline{\kappa}]$. Then, the two panels of Figure 1 show how $\Phi(\kappa)$ and $\Psi(\kappa)$ (defined in (21) and (22)) vary with $\kappa \in [\underline{\kappa}, \overline{\kappa}]$. As can be seen from the left panel, while $\Phi(\kappa)$ is a function, the part associated with the response of the economy to fundamental shocks, $\Psi(\kappa)$ is a set-valued map. The solid section of the curve corresponds to the case where $\psi(L)$ is invertible, and the dashed line corresponds to the case where $\psi(L)$ is non-invertible.

To highlight how the presence of sentiments can affect the response of the economy to fundamental shocks, we choose two typical equilibria from the set described above. In particular, we pick two values of κ which lie in the interval $[\underline{\kappa}, \overline{\kappa}]$. We will refer to the equilibrium corresponding to the lower of these two κ's as the low-κ equilibrium and the one associated with the higher value of κ as the large-κ equilibrium. The corresponding $\Phi(\kappa)$ and $\Psi(\kappa)$ are the dots in Figure 1. Next, we describe how the response of the economy in the two equilibria to fundamental and sentiment shocks can differ dramatically across equilibria.

The left panel shows the impulse response to the sentiment shock and the fundamental shock in the small-κ equilibrium are shown in the left panel of Figure 2. The right panel shows

[33] This implies that $g(L) = \frac{1}{1-\rho L}$.

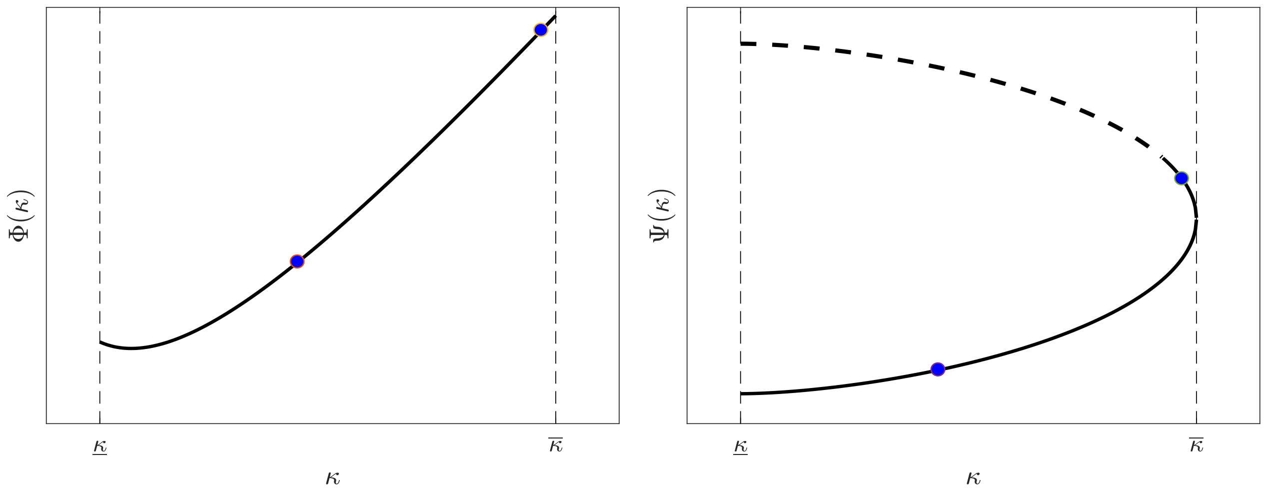


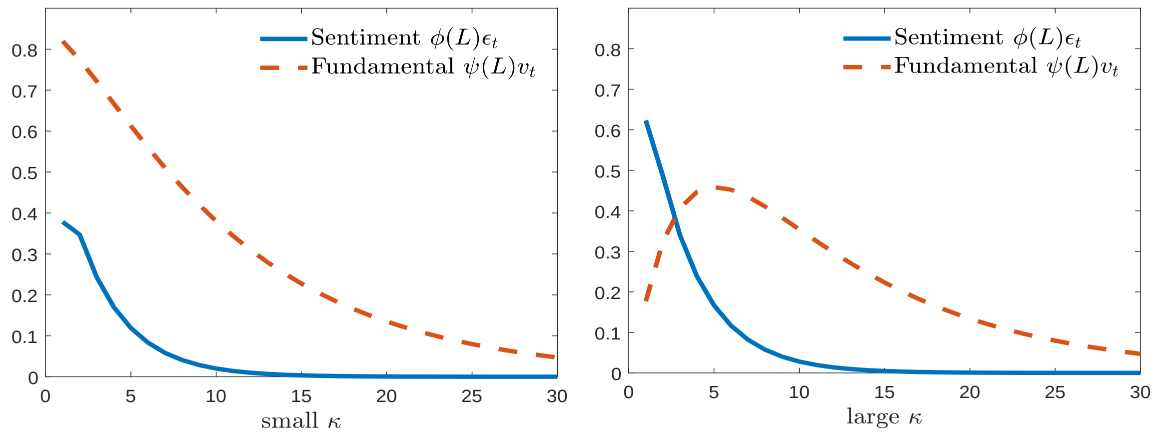
Figure 1: Coefficients of the Equilibrium Process

the impulse responses in the high-κ equilibrium. In the small-κ equilibrium, the response of the economy to changes in sentiments is less volatile than in the high-κ equilibrium (the blue solid lines in the two panels). Consequently, in the low-κ equilibrium, the aggregate outcome is mostly driven by changes in fundamentals. Since in this equilibrium, fundamentals account for the bulk of aggregate fluctuations, agents by observing aggregate outcomes can learn relatively a lot more about the realizations of the fundamental shock and as a result, the response to the fundamental shock is monotonic and closer to that in the fundamental equilibrium.

However, when κ is large, the sentiment part $\phi(L)\epsilon_t$ is more volatile and accounts for a larger fraction of aggregate fluctuations. This makes it more difficult for agents to infer information about the fundamental shock since most variations in aggregate outcomes are now driven by changes in sentiments. This results in a slower rate at which agents learn about the actual realizations of aggregate fundamental shocks. Consequently, when a fundamental shock actually hits the economy, agents are slow to realize this and this generates a hump-shaped rather than a monotonic response to the fundamental shock.

The hump-shaped response is an empirical regularity emphasized in the DSGE literature (see Christiano et al. (2005) and Smets and Wouters (2007) for example) in the context of many macroeconomic data series. One potential way to generate such a response is to model the presence of dispersed information. Since higher-order expectations are more anchored by the prior, they can explain the presence of additional inertia in an economy's response to shocks (See for example, Woodford (2002) and Angeletos and Huo (2018)). This paper presents another logically distinct way which could explain the presence of this additional inertia. In our model environment, the information is complete in the fundamental equilibrium and so there is no more dispersed information after agents observe aggregate outcomes. However, the introduction of the sentiment shock makes the information incomplete, and agents have to rely on Bayesian

Figure 2: Impulse Response in the Sentiment Equilibrium



learning to infer the shocks and others' action. Therefore, for a fixed κ, each individual behaves as if they face an exogenous signal process and their weighted average of fist-order and higher order beliefs about the fundamental as in Woodford (2002). However, unlike Woodford (2002), the range of κ and the structure and informativeness of signals in our model is endogenously determined as part of the equilibrium.

The effects of the sentiment shock in our model resembles that of the exogenous common noise in the dispersed information model such as Angeletos and La'O (2010). The presence of such noise in these models prevents agents from inferring the realizations of aggregate fundamentals and thus, alters the economy's response to changes in aggregate fundamentals. For example, in Angeletos and La'O (2010), the response to the exogenous common noise shock follows an AR(1) process and the response to the TFP shock follows an ARMA(2,1) process, which is almost identical to our model dynamics. The small/large κ sentiment equilibrium corresponds to the case in Angeletos and La'O (2010) where the variance of the common noise is relatively small/large.

Despite these apparent similarities, the environment studied in models such as Angeletos and La'O (2010) and our model are very different. Unlike in their setting, the sentient performs the role of a common noise term but crucially in *endogenous* in the sense that it is disciplined by equilibrium. More broadly, the sentiment equilibrium may be viewed as a particular microfoundation for these exogenous noise shocks that drive aggregate outcomes in the dispersed information literature.

Finally, providing a technical distinction between our model structure with that of Rondina and Walker (2014) is useful. The information structure in our model is similar to Rondina and Walker (2014), in the sense that there is a non-invertible component that prevents information from being fully revealed. However, there are four major differences between our model and theirs. First, unlike in their setup, in our model, the fundamental that drives the economy is always assumed to be invertible. In contrast we choose to model the the signal observed by agents as a weighted average of current and past fundamentals in such a way that the signal is non-invertible rather than the fundamental being non-invertible. Second, in Rondina and Walker (2014), because of the non-invertible fundamental, the response to the fundamental shock oscillates around its perfect information benchmark. In our model, due to the additional sentiment shock, the response to the fundamental shock is monotonic or hump-shaped, which is closer to the identified impulse response in the literature. Third, in the absence of shocks to sentiments, the fundamental equilibrium in our model is the same as the perfect information benchmark. In contrast, Rondina and Walker (2014) feature imperfect information about the shocks to aggregate fundamentals even in the fundamental equilibrium. Fourth and most importantly, Rondina and Walker (2014) do not explore the possibility of a sentiment equilibrium.

6 Conclusion

The objective of this paper was to establish whether endogenously arising sentiments could drive persistent aggregate fluctuations in the context of rational expectations equilibria. Within the class of the commonly used beauty contests, we showed that there exist a multiplicity of equilibria in which sentiments can drive aggregate fluctuations. The persistent sentiment equilibrium can arise in the absence of any fundamental shocks, and it can also add additional persistence to the response to fundamental shocks. Furthermore, we identified necessary and sufficient conditions under which these sentiments equilibria can account for persistent aggregate fluctuations, which do not depend on the private information agents might possess. This characterization serves as an guide for a growing literature in the field of macroeconomics that is trying to theoretically and quantitatively evaluate the importance of sentiments or correlated equilibria in trying to understand aggregate fluctuations.

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Appendix

A Restricting attention to equilibria with invertible $\phi(L)$ is without loss of generality

Proposition 4. *If $a_t = \boldsymbol{\psi}(L)\nu_t + \phi(L)\epsilon_t$ is a sentiment equilibrium, then $a_t = \boldsymbol{\psi}(L)\nu_t + \widetilde{\phi}(L)\epsilon_t$ is also a sentiment equilibrium where $\widetilde{\phi}(L)$ is given by:*

$$\widetilde{\phi}(L) = \phi(L) \prod_{n=0}^{N} \frac{L-\lambda_n}{1-\lambda_n L} \qquad \text{for any} \qquad N \in \mathbb{Z}_+ \text{ and } (\lambda_1, \lambda_2, \cdots, \lambda_N) \in (-1,1)^N$$

Furthermore, there exits a sentiment equilibrium $a_t = \boldsymbol{\psi}(L)\nu_t + \widetilde{\phi}(L)\epsilon_t$ such that

$$\phi(L)\phi(L^{-1}) = \widetilde{\phi}(L)\widetilde{\phi}(L^{-1}),$$

and $\widetilde{\phi}(L)$ is invertible.

B Static case with agg. fundamentals and sentiments

Agent i observes the signal:

$$x_i = \psi\theta + \phi\epsilon + \zeta_i$$

It follows that:

$$\mathbb{E}\left[\theta \mid x_i\right] = \frac{\psi\sigma_\theta^2}{\psi^2\sigma_\theta^2 + \phi^2 + \sigma_\zeta^2} x_i \qquad \text{and} \qquad \mathbb{E}\left[\epsilon \mid x_i\right] = \frac{\phi}{\psi^2\sigma_\theta^2 + \phi^2 + \sigma_\zeta^2} x_i$$

allora:

$$\mathbb{E}_i a = \psi\mathbb{E}\left[\theta \mid x_i\right] + \phi\mathbb{E}\left[\epsilon \mid x_i\right] = \frac{\psi^2\sigma_\theta^2 + \phi^2}{\psi^2\sigma_\theta^2 + \phi^2 + \sigma_\zeta^2} x_i$$

Using this in B.2 yields:

$$a_i = \frac{\gamma\psi^2\sigma_\theta^2 + \gamma\phi^2 + \varphi\psi\sigma_\theta^2}{\psi^2\sigma_\theta^2 + \phi^2 + \sigma_\zeta^2} x_i$$

Aggregating individual decisions and using the fact that $\int x_i di = a$, we get:

$$a = \int a_i di = \frac{\gamma\psi^2\sigma_\theta^2 + \gamma\phi^2 + \varphi\psi\sigma_\theta^2}{\psi^2\sigma_\theta^2 + \phi^2 + \sigma_\zeta^2} a$$

which requires that:

$$\left(\psi - \frac{\varphi}{2\left(1-\gamma\right)}\right)^2 + \frac{\phi^2}{\sigma_\theta^2} = r^2$$

where $r = \sqrt{\frac{1}{1-\gamma}\left(\frac{\varphi^2}{4(1-\gamma)} - \frac{\sigma_\zeta^2}{\sigma_\theta^2}\right)}$. The existence of sentiment equilibria requires that $r \geq 0$ which is guaranteed if the exogenous signal noise variance σ_ζ^2 is small enough. Importantly, sentiment equilibrium exist even when $\sigma_\zeta^2 = 0$.

C Proof of Propositions

C.1 Proof of Proposition 1

Consider an impulse response of the signals to an ϵ_t shock, where $\epsilon_0 = 1$, and $\epsilon_t = 0$ for $t \neq 0$. Note that for the sentiment process $\phi(L) = \sum_{t=0}^{\infty} \phi_t L^t$, ϕ_t is the same as the response of a_t at time t. To show that $\phi(L) = \sum_{t=0}^{k} L^t \phi_t L^t$, it is sufficient to show that the impulse response of a_t is zero from period k.

By Proposition 4, we only need to consider the case where $\phi(L)$ is invertible. If Assumption 1 and 2 are satisfied, $\mathbb{E}[\boldsymbol{\nu}_{t-\tau} \mid \mathbf{y}_i^t] = \boldsymbol{\nu}_{t-\tau}$ and agents also observe $a_{t-\tau} = \boldsymbol{\psi}(L)\boldsymbol{\nu}_{t-\tau} + \phi(L)\epsilon_{t-\tau}$ for $\tau \geq k$. As a result, agents observe $\phi(L)\epsilon_{t-\tau}$ perfectly for $\tau \geq k$. Because $\phi(L)$ is invertible, past sentiment shocks $\{\epsilon_{t-\tau}\}_{\tau=k}^{\infty}$ can be inferred perfectly. Particularly, when $t = k$, agents can infer $\{\epsilon_\tau\}_{\tau=-\infty}^{0}$ perfectly, and they know $\epsilon_0 = 1$ without uncertainty.

Recall that the signal process is given by

$$\begin{aligned} \mathbf{y}_{i,t} &= \mathbf{P}(L)\boldsymbol{\nu}_t + \mathbf{Q}(L)\boldsymbol{\zeta}_{i,t} \\ \mathbf{x}_{i,t} &= \mathbf{A}(L)a_t + \mathbf{B}(L)\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}, \end{aligned}$$

In the impulse response experiment, only $\epsilon_0 = 1$, and all other shocks are muted. Effectively, agents observe

$$\begin{aligned} \mathbf{y}_{i,t} &= \mathbf{0}, \\ \mathbf{x}_{i,t} &= \mathbf{A}(L)\phi_t\epsilon_0, \end{aligned}$$

where $\phi_t = 0$ for $t < 0$.

With $t \geq k$, after subtracting the part $\mathbf{A}(L)\phi_t\epsilon_0$ which agents observe perfectly, the signals are all zero. It follows that the optimal forecasts for all other shocks have to be zero, $\mathbb{E}_{i,t}[\boldsymbol{\zeta}_{i,t-\tau}] = 0$, $\mathbb{E}_{i,t}[\boldsymbol{\nu}_{t-\tau}] = 0$, and $\mathbb{E}_{i,t}[\epsilon_{t-\tau}] = 0$ for $t > 0$ and $\tau \geq 0$.

Therefore, the impulse response with $t \geq k$ is given by

$$a_t = \phi_t = \int \alpha\mathbb{E}_{i,t}[\mathbf{h}(L)\mathbf{u}_{i,t}] + \varphi\mathbb{E}_{i,t}[\mathbf{g}(L)\mathbf{v}_t] + \gamma\mathbb{E}_{i,t}[\phi(L)\epsilon_t] = \gamma\phi_t\epsilon_0. \tag{24}$$

Given that $\gamma < 1$ and $\epsilon_0 = 1$, it has to be that $\phi_t = 0$ for $t > k$. It follows that $\phi(L) = \sum_{t=0}^{k} L^t\phi_t L^t$.

C.2 Proof of Proposition 2

In a linear rational expectations equilibrium with only sentiment shock, the aggregate outcome can be written as

$$a_t = \phi(L)\epsilon_t$$

The information process can be summarized as

$$\begin{bmatrix} x_{i,t}^1 \\ x_{i,t}^2 \end{bmatrix} = \begin{bmatrix} \phi(L) & 1 \\ (L-\lambda)\phi(L) & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ z_{i,t} \end{bmatrix} \qquad \Leftrightarrow \qquad \mathbf{X}_{i,t} = \mathbf{M}(L)\, e_{i,t}$$

Note that the determinant of $\mathbf{M}(z)$ is

$$\det[\mathbf{M}(z)] = (z-\lambda)\phi(z)$$

and there is one root inside the unit circle, 0 and λ. This mapping can also be represented by an observationally equivalent invertible representation of system

$$\mathbf{X}_{i,t} = \underbrace{\mathbf{M}(L)\,\boldsymbol{\Sigma}\mathbf{W}\mathbb{B}(L;\lambda)}_{\widetilde{\mathbf{M}}(L)}\underbrace{\mathbb{B}\left(L^{-1};\lambda\right)'\mathbf{W}'\boldsymbol{\Sigma}^{-1}e_{i,t}}_{\widetilde{e}_{i,t}}$$

where

$$\begin{aligned}
\mathbb{B}(L;\lambda) &= \begin{bmatrix} \frac{L^{-1}-\lambda}{1-\lambda L^{-1}} & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{W} &= \begin{bmatrix} \frac{\sigma}{\sqrt{\sigma^2+\phi(\lambda)^2}} & \frac{\phi(\lambda)}{\sqrt{\sigma^2+\phi(\lambda)^2}} \\ -\frac{\phi(\lambda)}{\sqrt{\sigma^2+\phi(\lambda)^2}} & \frac{\sigma\phi(\lambda)}{\sqrt{\sigma^2+\phi(\lambda)^2}} \end{bmatrix}
\end{aligned}$$

Using the Kolomogrov-Weiner projection formulas:

$$\begin{aligned}
\int \mathbb{E}_{i,t}[a_t] &= \phi(L) + \frac{\left(1-\lambda^2\right)\sigma^2\phi(\lambda)}{(1-\lambda L)\left(\sigma^2+\phi(\lambda)^2\right)} \\
\int \mathbb{E}_{i,t}[z_{i,t}] &= a_t - \int \mathbb{E}_{i,t}a_t
\end{aligned}$$

Also, recall that equilibrium must satisfy:

$$a_t = \alpha \int \mathbb{E}_{i,t}z_{i,t} + \gamma \int \mathbb{E}_{i,t}a_t$$

As a result, it must be the case that:

$$\phi(L) = \frac{\left(1-\lambda^2\right)\sigma^2\phi(\lambda)(\alpha-\gamma)}{(1-\gamma)(1-\lambda L)\left(\sigma^2+\phi(\lambda)^2\right)}$$

Evaluating at $L = \lambda$ leads to

$$\phi(\lambda) = \frac{\sigma^2\phi(\lambda)(\alpha-\gamma)}{(1-\gamma)\left(\sigma^2+\phi(\lambda)^2\right)}$$

Therefore, the sentiment process is given by

$$\phi(L) = \sigma(1-\lambda^2)\sqrt{\frac{\alpha-1}{1-\gamma}}\frac{1}{1-\lambda L}$$

C.3 Proof of Proposition 3

The linear rational expectation equilibrium with both sentiment shock and fundamental shock can be written as

$$a_t = \phi(L)\epsilon_t + \psi(L)v_t$$

The signal process can be summarized as

$$\begin{bmatrix} x_{i,t}^1 \\ x_{i,t}^2 \\ x_{i,t}^3 \end{bmatrix} = \begin{bmatrix} 1 & \phi(L) & \psi(L) \\ 0 & (L-\lambda)g(L) & 0 \\ 0 & L\phi(L) & L\psi(L) \end{bmatrix} \begin{bmatrix} \zeta_{i,t} \\ \epsilon_t \\ v_t \end{bmatrix} \qquad \Leftrightarrow \qquad \mathbf{X}_{i,t} = \mathbf{M}\left(L\right) e_{i,t}$$

Note that the determinant of $M(z)$ is

$$\det[\mathbf{M}(z)] = \sigma z g(z)(z-\lambda)\phi(z)$$

and there are two roots inside the unit circle. Note that we restrict $\phi(L)$ is invertible by the logic in Proposition 4. This mapping can also be represented by an observationally equivalent invertible representation of system

$$\mathbf{X}_{i,t} = \underbrace{\mathbf{M}\left(L\right)\mathbf{\Sigma}\mathbf{W}_1\mathbb{B}\left(L;\lambda\right)\mathbf{W}_2\mathbb{B}\left(L;0\right)}_{\widetilde{\mathbf{M}}(L)}\underbrace{\mathbb{B}\left(L^{-1};0\right)'\mathbf{W}_2'\mathbb{B}\left(L^{-1};\lambda\right)'\mathbf{W}_{1'}\mathbf{\Sigma}^{-1}e_{i,t}}_{\widetilde{e}_{i,t}}$$

where

$$\begin{aligned}
\mathbb{B}\left(L;\omega\right) &= \begin{bmatrix} \frac{L^{-1}-\lambda}{1-\omega L^{-1}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{W}_1 &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\phi_\lambda}{\sqrt{\phi_\lambda^2+\psi_\lambda^2}} & 0 & \sqrt{\frac{\psi_\lambda^2}{\phi_\lambda^2+\psi_\lambda^2}} \\ \frac{\psi_\lambda}{\sqrt{\phi_\lambda^2+\psi_\lambda^2}} & 0 & \frac{\phi_\lambda\sqrt{\frac{\psi_\lambda^2}{\phi_\lambda^2+\psi_\lambda^2}}}{\psi_\lambda} \end{bmatrix} \\
\mathbf{W}_2 &= \frac{1}{\sqrt{\left(\sigma^2+\phi_0^2\right)\phi_\lambda^2+\left(\lambda^2\sigma^2+\phi_0^2\right)\psi_\lambda^2}} \\
&\quad \begin{bmatrix} -\lambda\sigma\sqrt{\psi_\lambda^2} & \sqrt{\left(\sigma^2+\phi_0^2\right)\phi_\lambda^2+\phi_0^2\psi_\lambda^2} & 0 \\ -\frac{\phi_0\psi_\lambda\sqrt{\phi_\lambda^2+\psi_\lambda^2}}{\sqrt{\psi_\lambda^2}} & -\frac{\lambda\sigma\phi_0\psi_\lambda\sqrt{\phi_\lambda^2+\psi_\lambda^2}}{\sqrt{\left(\sigma^2+\phi_0^2\right)\phi_\lambda^2+\phi_0^2\psi_\lambda^2}} & \frac{\sigma\left(\sqrt{\phi_\lambda^2}\sqrt{\left(\sigma^2+\phi_0^2\right)\phi_\lambda^2+\left(\lambda^2\sigma^2+\phi_0^2\right)\psi_\lambda^2}\right)}{\sqrt{\left(\sigma^2+\phi_0^2\right)\phi_\lambda^2+\phi_0^2\psi_\lambda^2}} \\ \sigma\phi_\lambda & \frac{\lambda\sigma^2\phi_\lambda\sqrt{\psi_\lambda^2}}{\sqrt{\left(\sigma^2+\phi_0^2\right)\phi_\lambda^2+\phi_0^2\psi_\lambda^2}} & \frac{\phi_0\psi_\lambda\sqrt{\phi_\lambda^2}\sqrt{\phi_\lambda^2+\psi_\lambda^2}\sqrt{\left(\sigma^2+\phi_0^2\right)\phi_\lambda^2+\left(\lambda^2\sigma^2+\phi_0^2\right)\psi_\lambda^2}}{\phi_\lambda\sqrt{\psi_\lambda^2}\sqrt{\left(\sigma^2+\phi_0^2\right)\phi_\lambda^2+\phi_0^2\psi_\lambda^2}} \end{bmatrix}
\end{aligned}$$

Here, we use ϕ_0, ϕ_λ, and ψ_λ to denote $\phi(0), \phi(\lambda)$, and $\psi(\lambda)$.

Using the Kolomogrov-Weiner projection formulas:

$$\begin{aligned}
\int \mathbb{E}_{i,t}[a_t] &= \left(\phi(L) - \frac{\psi_\lambda^2\lambda\sigma^2\phi_0(\lambda-L)+\phi_\lambda^2(1-\lambda L)\sigma^2\phi_0}{(1-\lambda L)\left(\psi_\lambda^2\left(\lambda^2\sigma^2+\phi_0^2\right)+\phi_\lambda^2\left(\sigma^2+\phi_0^2\right)\right)}\right)\epsilon_t \\
&\quad + \left(\psi(L) - \frac{\left(1-\lambda^2\right)\sigma^2\phi_0\psi_\lambda\phi_\lambda}{(1-\lambda L)\left(\psi_\lambda^2\left(\lambda^2\sigma^2+\phi_0^2\right)+\phi_\lambda^2\left(\sigma^2+\phi_0^2\right)\right)}\right)v_t \\
\int \mathbb{E}_{i,t}[\theta_t] &= \frac{\left(1-\lambda^2\right)g(\lambda)\phi_\lambda\left(\psi_\lambda\left(\lambda\sigma^2 L-\phi_0^2\right)\epsilon_t-\phi_\lambda\left(\sigma^2+\phi_0^2\right)v_t\right)}{(1-\lambda L)\left(\psi_\lambda^2\left(\lambda^2\sigma^2+\phi_0^2\right)+\phi_\lambda^2\left(\sigma^2+\phi_0^2\right)\right)} + g(L)v_t
\end{aligned}$$

Also, recall that equilibrium must satisfy:

$$a_t = \varphi \int \mathbb{E}_{i,t}[\theta_t] + \gamma \int \mathbb{E}_{i,t}[a_t]$$

As a result, it must be the case that:

$$\phi(L) = \left(\phi_0 + \frac{\lambda\sigma^2}{(1-\gamma)} \frac{\gamma\phi_0(\phi_\lambda^2+\psi_\lambda^2) + \varphi\left(1-\lambda^2\right) g_\lambda \psi_\lambda \phi_\lambda}{\psi_\lambda^2\left(\lambda^2\sigma^2+\phi_0^2\right)+\phi_\lambda^2\left(\sigma^2+\phi_0^2\right)} L\right) \frac{1}{1-\lambda L} \tag{25}$$

$$\psi(L) = \frac{\varphi}{1-\gamma} g(L) - \frac{\left(1-\lambda^2\right)\phi_\lambda}{(1-\gamma)} \frac{\gamma\sigma^2\phi_0\psi_\lambda + \varphi g_\lambda \phi_\lambda\left(\sigma^2+\phi_0^2\right)}{\psi_\lambda^2\left(\lambda^2\sigma^2+\phi_0^2\right)+\phi_\lambda^2\left(\sigma^2+\phi_0^2\right)} \frac{1}{1-\lambda L} \tag{26}$$

with $\phi(\lambda)$ and $\psi(\lambda)$ satisfying

$$\begin{aligned}
\frac{\varphi\left(1-\lambda^2\right)\phi_0 g_\lambda \psi_\lambda \phi_\lambda - \gamma\sigma^2(\phi_\lambda^2+\lambda^2\psi_\lambda^2)}{\psi_\lambda^2\left(\lambda^2\sigma^2+\phi_0^2\right)+\phi_\lambda^2\left(\sigma^2+\phi_0^2\right)} &= (1-\gamma) \\
\frac{(1-\gamma)g_\lambda(\sigma^2\lambda^2+\phi_0^2)\psi_\lambda - \sigma^2\gamma\phi_0\phi_\lambda}{\psi_\lambda^2\left(\lambda^2\sigma^2+\phi_0^2\right)+\phi_\lambda^2\left(\sigma^2+\phi_0^2\right)} &= (1-\gamma)
\end{aligned}$$

Given ϕ_0, we can solve for ϕ_λ and ψ_λ as

$$\begin{aligned}
\phi_\lambda &= \frac{\lambda^2\sigma^2+(1-\gamma)\phi_0^2}{(1-\gamma)\left(1-\lambda^2\right)\phi_0} \\
\psi_\lambda &= \frac{\varphi\left(1-\lambda^2\right)\phi_0 \pm \sqrt{\left(\varphi\left(1-\lambda^2\right)\phi_0 g_\lambda\right)^2 - 4\left(\sigma^2+(1-\gamma)\phi_0^2\right)\left(\lambda^2\sigma^2+(1-\gamma)\phi_0^2\right)}}{2\phi_0(1-\gamma)\left(1-\lambda^2\right)}
\end{aligned}$$

Denote ϕ_0 as κ. To make sure that ψ_λ is well defined, it has to be that

$$\left(\varphi\left(1-\lambda^2\right)g(\lambda)\right)^2\kappa^2 > 4\left(\sigma^2+(1-\gamma)\kappa^2\right)\left(\lambda^2\sigma^2+(1-\gamma)\kappa^2\right)$$

Equation (25) and (26) can be easily written in terms of $\Phi(\kappa)$ and $\Psi(\kappa)$ accordingly.

Another restriction for κ is that $\Phi(\kappa)$ has to make sure that $\phi(L)$ is invertible, but this is a further selection among the equilibria that satisfying the condition above.

C.4 Proof of Proposition 4

Assume that the aggregate action a_t is given by

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t$$

Let m, n, r, ℓ denote the dimensions of exogenous signals, endogenous signals, exogenous aggregate shocks, and exogenous idiosyncratic shocks, respectively. Agents' information structure specified in Section 2 can be represented by the following matrix

$$\mathbf{s}_{i,t} \equiv \begin{bmatrix} y_{i,t}^1 \\ \vdots \\ y_{i,t}^m \\ x_{i,t}^1 \\ \vdots \\ x_{i,t}^n \end{bmatrix} = \begin{bmatrix} 0 & P_{11}(L) & \dots & P_{1r}(L) & Q_{11}(L) & \dots & Q_{1\ell}(L) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & P_{m1}(L) & \dots & P_{mr}(L) & Q_{m1}(L) & \dots & Q_{m\ell}(L) \\ A_1(L)\phi(L) & A_1(L)\psi_1(L) + B_{11}(L) & \dots & A_1(L)\psi_r(L) + B_{1r}(L) & C_{11}(L) & \dots & C_{1\ell}(L) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ A_n(L)\phi(L) & A_n(L)\psi_1(L) + B_{n1}(L) & \dots & A_n(L)\psi_r(L) + B_{nr}(L) & C_{n1}(L) & \dots & C_{n\ell}(L) \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \nu_t^1 \\ \vdots \\ \nu_t^r \\ \zeta_{i,t}^1 \\ \vdots \\ \zeta_{i,t}^\ell \end{bmatrix}$$

More compactly, we can represent the information structure by

$$\mathbf{s}_{i,t} = \begin{bmatrix} \mathbf{y}_{i,t} \\ \mathbf{x}_{i,t} \end{bmatrix} = \mathbf{M}(L) \begin{bmatrix} \epsilon_t \\ \boldsymbol{\nu}_t \\ \boldsymbol{\zeta}_{i,t} \end{bmatrix}.$$

Suppose $\widetilde{\phi}(L)$ satisfies that

$$\widetilde{\phi}(L)\widetilde{\phi}(L^{-1}) = \phi(L)\phi(L^{-1}).$$

Denote $\widetilde{\mathbf{M}}(L)$ as the matrix where $\phi(L)$ is replaced by $\widetilde{\phi}(L)$. This replacement implies that agent i believes that the process for sentiments follow $\widetilde{\phi}(L)$ instead of $\phi(L)$. We will show that the forecast rules based on $\mathbf{M}(L)$ is the same as those based on $\widetilde{\mathbf{M}}(L)$. Note that

$$\widetilde{\mathbf{M}}(L)\widetilde{\mathbf{M}}'(L^{-1}) = \mathbf{M}(L)\mathbf{M}'(L^{-1}).$$

This equality implies that the fundamental representation of $\mathbf{M}(L)$ and $\widetilde{\mathbf{M}}(L)$ is the same. Denoting the fundamental representation of $\mathbf{M}(L)$ as $\mathbf{B}(L)$, we have

$$\mathbf{M}(L)\mathbf{M}'(L^{-1}) = \widetilde{\mathbf{M}}(L)\widetilde{\mathbf{M}}'(L^{-1}) = \mathbf{B}(L)\mathbf{B}'(L^{-1}),$$

where $\mathbf{B}(L)$ is invertible. For any stochastic variable

$$f_{i,t} = \mathbf{F}(L) \begin{bmatrix} \epsilon_t \\ \boldsymbol{\nu}_t \\ \mathbf{u}_{i,t} \end{bmatrix},$$

the Wiener-Hopf prediction formula using $\mathbf{M}(L)$ is given by

$$\mathbb{E}_{i,t}[f_{i,t}] = \left[\mathbf{F}(L)\mathbf{M}'(L^{-1})\mathbf{B}'(L^{-1})\right]_+ \mathbf{B}(L)^{-1}\mathbf{s}_{i,t},$$

and the forecasting rule using $\widetilde{\mathbf{M}}(L)$ is given by

$$\widetilde{\mathbb{E}}_{i,t}[f_{i,t}] = \left[\mathbf{F}(L)\widetilde{\mathbf{M}}'(L^{-1})\mathbf{B}'(L^{-1})\right]_+ \mathbf{B}(L)^{-1}\mathbf{s}_{i,t}.$$

Supposing an agent wants to forecast a stochastic variable driven by exogenous aggregate or idiosyncratic shocks, we have

$$\mathbf{F}(L) = \left[0, F_2(L), F_3(L) \dots, , F_{r+\ell+1}(L)\right].$$

It is straightforward to verify that

$$\mathbf{F}(L)\mathbf{M}'(L^{-1}) = \mathbf{F}(L)\widetilde{\mathbf{M}}'(L^{-1}).$$

As a result, $\mathbb{E}_{i,t}[f_{i,t}] = \widetilde{\mathbb{E}}_{i,t}[f_{i,t}]$.

Suppose an agent wants to forecast the aggregate action, i.e., $f_{i,t} = a_t$. If the agent believes that $a_t = \phi(L)\epsilon_t$, then

$$\mathbf{F}(L) = \left[\phi(L), 0, \ldots, 0\right].$$

Similarly, if the agent believes $a_t = \widetilde{\phi}(L)\epsilon_t$, then

$$\mathbf{F}(L) = \left[\widetilde{\phi}(L), 0, \ldots, 0\right].$$

Due to that $\widetilde{\phi}(L)\widetilde{\phi}(L^{-1}) = \phi(L)\phi(L^{-1})$, the following identity holds

$$\left[\phi(L), 0, \ldots, 0\right]\mathbf{M}'(L^{-1}) = \left[\widetilde{\phi}(L), 0, \ldots, 0\right]\widetilde{\mathbf{M}}'(L^{-1}),$$

which implies that $\mathbb{E}_{i,t}[a_t] = \widetilde{\mathbb{E}}_{i,t}[a_t]$. Because the inferences are the same under the two specifications of the sentiment processes, both of them will be REE.

Assume that $a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t$ is an equilibrium. By the Wold representation theorem, there always exists $\widetilde{\phi}(L)$ and $w_t \sim \mathcal{N}(0,1)$ such that

$$\phi(L)\epsilon_t = \widetilde{\phi}(L)w_t,$$

and $\widetilde{\phi}(L)$ is invertible. By construction, it follows that

$$\phi(L)\phi(L^{-1}) = \widetilde{\phi}(L)\widetilde{\phi}(L^{-1}).$$

Consider the signal $\widetilde{\mathbf{s}}_{i,t} = \begin{bmatrix}\mathbf{y}_{i,t} & \widetilde{\mathbf{x}}_{i,t}\end{bmatrix}$. The endogenous signal is generated by

$$\widetilde{\mathbf{x}}_{i,t} = \mathbf{A}(L)(\boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \widetilde{\phi}(L)w_t) + \mathbf{B}(L)\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}.$$

While in the original equilibrium, the endogenous signal is generated by

$$\widetilde{\mathbf{x}}_{i,t} = \mathbf{A}(L)(\boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t) + \mathbf{B}(L)\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}.$$

The autocorrelation-generating function of $\mathbf{s}_{i,t}$ and $\widetilde{\mathbf{s}}_{i,t}$ are the same. The sentiment part $\phi(L)\epsilon_t$ and $\widetilde{\phi}(L)w_t$ are identical. Therefore, $a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \widetilde{\phi}(L)w_t$ is also an equilibrium.

D Examples in Section 4

D.1 Agents do not observe past aggregate actions

Consider the following environment: the idiosyncratic fundamental $z_{i,t}$ is given by a AR(1):

$$z_{i,t} = \frac{1}{1-\rho L}u_{i,t}$$

Agents receive two signals every period: (1) two periods before aggregate action a_{t-2}; (2) noisy signal $x_{i,t}$ about current aggregate action:

$$x_{i,t} = a_t + u_{i,t}$$

where

$$a_t = \phi(L)\epsilon_t$$

An educated guess for the equilibrium path of aggregate action is:

$$\phi(L) = \phi_0 + \phi_1 L$$

Given this guess, the problem can be transformed into a static problem with the relevant information encoded in the following modified signals:

$$\begin{aligned} w_{i,t}^1 &= \phi_0 \epsilon_{t-1} + u_{i,t-1} \\ w_{i,t}^2 &= (\phi_0 \epsilon_t + \phi_1 \epsilon_{t-1}) + u_{i,t} \end{aligned}$$

The covariance matrix of $w_{i,t} = [w_{i,t}^1, w_{i,t}^2]'$ can be written as:

$$\Omega = \begin{bmatrix} \phi_0^2 + \sigma_u^2 & \psi_0\psi_1 \\ \psi_0\psi_1 & (\phi_0^2 + \phi_1^2) + \sigma_u^2 \end{bmatrix}$$

Then using the Kalman filter, any equilibrium satisfies:

$$\begin{aligned} \phi_0 &= \pm\sigma_u \sqrt{\frac{(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\alpha^2\rho^2)}}{1-\gamma}} \\ \phi_1 &= \phi_0 \frac{(\alpha-1) \pm \sqrt{((\alpha-1)^2 - 4\alpha^2\rho^2)}}{2\alpha\rho} \end{aligned}$$

D.2 Agents do not observe lagged innovations to aggregate fundamentals

Agents receive three signals every period: (1) exogenous signal about the past value of common aggregate fundamental $y_t = \nu_{t-2}$; (2) an endogenous signal about aggregate action

$$x_{i,t} = a_t + \zeta_{i,t},$$

(3) and last period aggregate action a_{t-1}. This information structure implies that last period aggregate fundamental ν_{t-1} is not directly observable, and an educated guess is that

$$\begin{aligned} \phi(L) &= \phi_0 + \phi_1 L \\ \psi(L) &= \psi_0 + \psi_1 L \end{aligned}$$

The equivalent signal process is

$$
\begin{aligned}
w_{i,t}^1 &= \phi_0 \epsilon_{t-1} + \psi_0 v_{t-1} \\
w_{i,t}^2 &= (\phi_0 \epsilon_t + \phi_1 \epsilon_{t-1} + \psi_0 v_t + \psi_1 v_{t-1}) + \zeta_{i,t}
\end{aligned}
$$

Indeed, we can verify that the set of persistent sentiment equilibria is

$$
\begin{aligned}
\phi_1 &= \pm \sigma_v^2 \psi_0 \sqrt{\frac{\varphi \sigma_v^2 \psi_0 - \sigma_u^2 - (1-\gamma)\left(\phi_0^2 + \sigma_v^2 \psi_0^2\right)}{(1-\gamma)\sigma_v^2\left(\phi_0^2 + \sigma_v^2 \psi_0^2\right)}} \\
\psi_1 &= \mp \phi_0 \sqrt{\frac{\varphi \sigma_v^2 \psi_0 - \sigma_u^2 - (1-\gamma)\left(\phi_0^2 + \sigma_v^2 \psi_0^2\right)}{(1-\gamma)\sigma_v^2\left(\phi_0^2 + \sigma_v^2 \psi_0^2\right)}}
\end{aligned}
$$

Note that ϕ_0 and ψ_0 have to satisfy

$$
\frac{\varphi \sigma_v^2 \psi_0 - \sigma_u^2}{1-\gamma} > \phi_0^2 + \sigma_v^2 \psi_0^2
$$

and

$$
\left| \frac{\phi_1}{\phi_0} \right| < 1
$$

In equilibrium

$$
\begin{aligned}
\phi_0^2 + \psi_0^2 \sigma_v^2 + \phi_1^2 + \psi_1^2 \sigma_v^2 &= \frac{\varphi \sigma_v^2 \psi_0 - \sigma_u^2}{1-\gamma} \\
\phi_0 \phi_1 &= -\sigma_v^2 \psi_0 \psi_1
\end{aligned}
$$

E Details on Section 5

The signal is

$$
x_t = (L-\lambda)\phi(L)\epsilon_t
$$

The fundamental representation is

$$
x_t = \phi(L)(1-\lambda L)\frac{L-\lambda}{1-\lambda L}\epsilon_t \equiv \phi(L)(1-\lambda L) w_t
$$

The forecast about $L^k \phi(L)\epsilon_t$ is

$$
\begin{aligned}
&\mathbb{E}_t[L^k \phi(L)\epsilon_t] \\
=&\mathbb{E}_t\left[L^k \phi(L)\phi(L^{-1})(L^{-1}-\lambda)\frac{1}{\phi(L^{-1})(1-\lambda L^{-1})}\right]_+ \frac{1}{\phi(L)(1-\lambda L)}(L-\lambda)\phi(L)\epsilon_t \\
=&\mathbb{E}_t\left[L^k \phi(L)\frac{1-\lambda L}{L-\lambda}\right]_+ \frac{L-\lambda}{(1-\lambda L)}\epsilon_t \\
=&\mathbb{E}_t\left[L^k \phi(L)\frac{1-\lambda L}{L-\lambda} - \lambda^k \phi(\lambda)\frac{1-\lambda^2}{L-\lambda}\right] \frac{L-\lambda}{(1-\lambda L)}\epsilon_t \\
=&L^k \phi(L)\epsilon_t - \lambda^k \phi(\lambda)\frac{1-\lambda^2}{1-\lambda L}\epsilon_t
\end{aligned}
$$

The forecast error is then given by

$$\lambda^k \phi(\lambda) \frac{1-\lambda^2}{1-\lambda L} \epsilon_t$$