

# Investment, Liquidity, and Financing under Uncertainty\*

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## Abstract

We develop a model of investment under uncertainty for a firm facing external financing costs. Such a firm prefers to fund its investment through internal funds, so that the firm's optimal investment policy and value now depend on the size of its retained earnings. We show that the standard real options results are significantly modified when there are external financing costs. Importantly the investment hurdle is highly non-monotonic in the firm's internal funds: when these are sufficient to cover capital expenditures then the investment hurdle is decreasing in the size of internal funds. But when they fall short then the firm's investment policy becomes more and more conservative when it accumulates cash, as it has stronger incentives to postpone its investment until the point where it has sufficient internal funds to entirely cover its investment outlays. With multiple rounds of options, a financially constrained firm may choose to over-invest in order to mitigate under-investment problems in the future due to financial constraints. Our analysis brings out the subtle interactions between sources of funds (external, internal, and prospective retained earnings once the investment is undertaken) and the optimal timing of investment.

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# 1 Introduction

In their influential textbook Dixit and Pindyck (1994) condense the essence of investment decisions to three key attributes: i) the degree of *irreversibility*; ii) the *risk* over future revenue; and, iii) the flexibility in the *timing* of the decision. In this paper we add a fourth attribute: the funding *cost* of the investment. Essentially all the theory of investment under uncertainty following McDonald and Siegel (1986) assumes that firms operate in frictionless capital markets. This is for good reason, as the firm's investment decision can then be formulated as a simple *real option problem* involving the optimal exercise and valuation of an American option. All the option pricing tools developed by Black and Scholes (1973) and Merton (1973) can then be deployed to analyze the real options problem.

An important drawback of this approach, however, is that firms in practice do not operate in frictionless capital markets. They face significant external financing costs and as a result rely mostly on internally generated funds to finance their investments. The recent financial crisis is an important reminder of how severe external financing costs can be in extreme situations and how much they can affect corporate investment and the macroeconomy. The obvious theoretical questions then are how corporate investment decisions and the valuation of investments under uncertainty are affected by external financing costs and the size of retained earnings. These are the questions addressed in this paper.

Although the classical tools of option pricing theory can no longer be directly applied (and although the analysis of the investment problem and its financing involves solving a considerably more involved two-dimensional partial differential equation) the results we obtain are intuitive and striking. First, both the growth options in the start-up phase and the abandonment options in the mature phase of the firm's life-cycle are worth less when the firm faces external financing constraints. Second, the *hurdle* for abandoning an asset is higher when the firm faces external financing costs: A mature firm operating an asset that is losing money is more likely to abandon this asset when it runs out of cash than a firm operating in a frictionless capital market. The reason is that the firm facing external financing costs is capitalizing expected future external financing costs and setting these against the present value of the asset. However, should the financially constrained firm decide to continue operating the asset when it runs out of cash by raising external funds, the firm will tend to raise *more* external funds the *higher* are external financing costs, as the firm seeks to limit the risk of having to return to capital markets again in the future. Note that this remarkable result can only be obtained in a dynamic model rather than a static one.

Third, in the start-up phase the firm with external financing costs will also have a higher hurdle for investment. Moreover, and most remarkably, this hurdle is a non-monotonic function of the firm's liquid assets: When the firm's internal funds are sufficient to entirely cover the costs of the investment then the firm's hurdle for investment is lower the higher the firm's internal funds. In contrast, when the firm's internal funds cannot cover the entire cost of the investment then the hurdle may be sharply increasing with the firm's internal funds. The reason is that when the firm is close to being able to entirely fund its investment with retained earnings it has a strong incentive to delay investment until it has sufficient

funds to be able to entirely avoid tapping costly external funds. This is in our view the most striking effect of the presence of external financing costs. An important implication of this result is that investment is not necessarily more likely when the firm has more cash. Investment could well be delayed further, as the firm's priority becomes avoiding reliance on external funds. In sum, for firms facing external financing costs the value of an investment opportunity is not just tied to *timing optionality* but also to *flexibility*, which depends on both the financial ability to seize an investment opportunity and on timing optionality.

An important implication of the results that hurdles for both abandonment and growth options are higher for firms facing external financing costs is that the frequency at which firms invest over time is likely to be lower when firms face higher external financing costs, as Whited (2006) finds. Similarly, another implication of the model that is consistent with the findings of DeAngelo, DeAngelo, Stulz (2010), is that firms facing external financing costs will only raise new funds by tapping equity markets when they need them, either because they have a valuable investment opportunity they cannot cover with internal funds or because they are burning cash and need new funds to be able to survive.

**Related literature.** Following Brennan and Schwartz (1985) and McDonald and Siegel (1986) the basic formulation of the investment under uncertainty problem has been extended in many different directions. Majd and Pindyck (1987) enrich the analysis with a time-to-build feature. Dixit (1989) uses the real option approach to examine entry and exit from a productive activity. Titman (1985) and Williams (1991) analyze real estate development in a real options framework. Abel and Eberly (1994) analyze a unified framework of investment under uncertainty that integrates the  $q$  theory of investment with the real options approach. Grenadier (1996, 2002) and Lambrecht and Perraudin (2003) extend the real options decision problem to a game-theoretic environment. Grenadier and Wang (2005) incorporate informational asymmetries and agency problems into the real options framework. Grenadier and Wang (2007) study the impact of time-inconsistent hyperbolic discounting on real-option exercising strategies.

Boyle and Guthrie (2003) is the first study of real options in the presence of financial constraints. In their model the firm can only pledge a fraction of its value, which constrains its ability to fund investments. As in our model, the firm's hurdle depends on its accumulated internal funds. However, unlike in our model the firm can continue operations even when it has arbitrarily negative internal funds and hence the firm's value can be arbitrarily negative and could not be abandoned, which is counterfactual. Sundaresan, Wang, and Yang (2013) study the optimal dynamic capital structure choice for a firm as it goes through its life-cycle by optimally choosing the exercising timing for a collection of sequentially ordered growth options and converting them into assets in place. Mauer and Triantis (1994) consider a real options problem for a levered firm, which may face recapitalization costs when its operating performance is poor. However, as in Leland (1994) the firm does not otherwise incur any external financing costs. In their model the levered firm has a lower hurdle as it seeks to bring tax-shield benefits of debt financing forward in time. Décamps and Villeneuve (2007) consider a financially constrained firm with an asset in place generating cash-flows that are subject to i.i.d shocks and a growth option, which raises the drift of the cash-flow process and

can only be financed with internal funds. They characterize the firm's optimal investment and dividend policy. Asvanunt, Broadie and Sundaresan (2007) also consider a real options problem for a levered firm. Unlike Mauer and Triantis (1994) and Leland (1994), however, they also introduce external equity financing costs in the form of dilution costs and allow the firm to accumulate internal funds. As in Décamps and Villeneuve (2007), they characterize the firm's optimal investment and payout policy (together with optimal leverage) and show that due to the external equity financing costs the hurdle for investment for the levered firm can be higher than for a financially unconstrained firm.

Our model is also related to the recent literature on dynamic corporate financial models with financial constraints. In particular, Décamps, Mariotti, Rochet, and Villeneuve (2011) and Bolton, Chen, and Wang (2011, 2013, 2014). Décamps, Mariotti, Rochet, and Villeneuve (2011) characterize the optimal payout and equity issuance policy of a firm facing external financing costs. Bolton, Chen, and Wang (2011) develops a  $q$ -theory model of investment for a firm facing external financing costs. Bolton, Chen, and Wang (2013) considers the optimal timing of equity issuance, payout and investment in a  $q$ -theory of investment setting where external financing costs are stochastic. Bolton, Chen, and Wang (2014) augments Décamps, Mariotti, Rochet, and Villeneuve (2011) by allowing the firm to issue term debt and considering a dynamic tax tradeoff theory for firms facing external financing costs. A major simplifying assumption in this latter literature is that cash-flow shocks are *transitory* i.i.d. shocks.

Our work is also related to two other sets of dynamic models of financing. First, DeMarzo, Fishman, He, and Wang (2012) develop a dynamic contracting model of corporate investment and financing with managerial agency, by building on Bolton and Scharfstein (1990) and using the dynamic contracting framework of DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007). These models derive optimal dynamic contracts and corporate investment with capital adjustment costs. Second, Rampini and Viswanathan (2010, 2011) develop dynamic models of investment with financing constraints, in which the firm is subject to endogenous collateral constraints induced by limited enforcement. Finally, our model also relates to the incomplete-markets real-options framework of Miao and Wang (2007), who consider a risk-averse decision maker holding an illiquid option in an incomplete markets setting. They show that the standard positive effect of volatility on option value can be offset by the agent's precautionary savings motive.

More broadly, our analysis in this paper contributes to the literature on financial frictions and corporate investment in the vein of Fazzari, Hubbard, and Petersen (1988), Froot, Scharfstein, and Stein (1993), and Kaplan and Zingales (1997). The first-generation models in this literature are purely static and thus not set up to study the optionality of investment and financing decisions. More recent models of investment with financial constraints, including Gomes (2001), Hennessy and Whited (2005, 2007), Riddick and Whited (2009) are dynamic models, but they also do not focus on the *joint* real option exercise and liquidity hoarding decisions.

## 2 Model

**Operating revenues and profits.** We consider a firm with an investment opportunity modeled as in McDonald and Siegel (1986). At any point in time  $t \geq 0$ , the firm can exercise an investment opportunity by paying a fixed investment outlay  $I > 0$ . Upon exercising, the firm then immediately obtains a perpetual stream of non-negative stochastic revenue  $Y_t$ . We assume that  $Y$  follows a geometric Brownian motion (GBM) process:

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t^Y, \quad (1)$$

where  $\mu$  is the drift parameter,  $\sigma$  the volatility parameter, and  $B^Y$  is a standard Brownian motion. Once it has exercised its growth option, the firm also incurs a constant flow operating cost  $Z > 0$  to operate the asset. The operating profit is then given by

$$Y_t - Z \quad (2)$$

per unit of time for as long as it continues operating the project. Should the firm deem that it is no longer worth continuing the operation, it can stop the project and there will be no scrap value. The firm has an American-style liquidation option where the timing of the option exercising decision is endogenously chosen.

Before undertaking the investment, the firm does not incur any costs, and while the firm is waiting to invest, the revenue process  $Y_t$  continues to evolve according to (1). In sum, the simplest formulation of the life-cycle of the firm in our model allows for three phases: a start-up phase, a mature phase, and a liquidation (a strong form of scale-down) phase.

We assume that investors are risk neutral, so that all cash flows are discounted at the risk-free rate,  $r$ . Equivalently, we may interpret that the revenue generating process (1) has already captured the risk adjustment, i.e., under the risk-neutral measure and hence we may use the risk-free rate to discount the firm's profits. One may pursue the risk-neutral measure interpretation when analyzing a firm's risk-return tradeoff.

**Liquidity hoarding.** For a financially constrained firm, the key is the cash accumulation dynamics. We next discuss cash accumulation in both phases.

*The startup phase.* At the beginning of the start-up phase ( $t = 0$ ) the firm is endowed with a stock of cash (or, more generally, a liquidity hoard comprising both cash and marketable securities) of  $W_0$ . As long as the firm does not spend this cash it simply accumulates liquid wealth at the risk-free rate  $r$  as follows:

$$dW_t = rW_t dt, \quad t \geq 0. \quad (3)$$

Note that since the firm earns the risk-free rate  $r$  on its cash it does not need to pay out any cash to its shareholders, and shareholders weakly prefer hoarding cash inside the firm. If the firm were to earn less than the risk-free rate on its cash, it would also face an optimal payout decision. For simplicity we do not consider this generalization of the model.<sup>1</sup> As the

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<sup>1</sup>For dynamic models with cash-carrying costs, see Bolton, Chen, and Wang (2011) and DéCamps, Martiotti, Rochet, and Villeneuve (2011) for example.

firm incurs no cost in carrying cash, without loss of generality, the firm never pays out its cash as long as it operates the asset.

When the firm's liquidity  $W$  is insufficient to cover investment costs, i.e.  $W < I$ , obviously the firm will have to raise external financing or continue to accumulate internal funds in order to finance the cost of exercising the growth option. Note that the firm will also need funds to finance potential operating losses after the growth option is exercised.

We introduce the standard specification for the external financing costs as follows: if the firm needs external funds  $F$  net of fees, it incurs an external financing cost  $\Phi(F)$ . Hence, the firm must raise a gross amount  $F + \Phi(F)$ . For simplicity, we assume that the equity issuance cost function for external financing takes the following form:

$$\Phi(F) = \phi_0 + \phi_1 F, \quad (4)$$

where  $\phi_0 > 0$  is the fixed cost parameter and  $\phi_1 \geq 0$  is the marginal cost of external financing. Intuitively, when the fixed equity issuance cost is sufficiently high, the firm prefers liquidation over equity issuance. While in theory, we may allow for different equity issuance cost functions in the start-up and mature phases to capture different degrees of financing frictions (e.g., agency costs and informational frictions) in the start-up and mature phases, we keep the functional forms to be the same in both phases for simplicity.

*The mature phase.* During the mature phase (after exercising its growth option) the firm's liquidity  $W_t$  accumulates as follows:

$$dW_t = (rW_t + Y_t - Z)dt + dC_t, \quad W_t \geq 0. \quad (5)$$

The first term in (5) denotes the firm's internal funds  $W_t$  (which earn the interest rate  $r$ ) plus the operating revenue  $Y_t$  minus the operating cost  $Z$ . The second term in (5),  $dC_t$ , denotes the net external funds that the firm chooses to raise through an external equity issue. We assume that the equity issuance cost function in this phase is also given by (4).

Note that the firm weakly never pays out and hence  $dC_t \geq 0$ . Total profits include interest income  $rW$  and operating profits  $Y_t - Z$ . Without external financing (i.e.,  $dC_t = 0$ ), profits increase liquidity hoard  $W$ , i.e. when  $rW_t + Y_t - Z > 0$ , and liquidity hoard finances losses when  $rW_t + Y_t - Z < 0$ . Finally, after the firm has chosen to liquidate/scale-down its investment, it simply pays out its remaining cash  $W_t$  to shareholders and closes down.

The firm's dynamic optimization problem thus involves a sequence of two optimal stopping decisions: an investment timing decision followed by an abandonment timing decision. Importantly, liquidity plays a critical role in both phases. Before providing the solution for a financially constrained firm's optimization problem, we first summarize the main results for a financially unconstrained firm.

### 3 The first-best benchmark

In the perfect capital markets world, where the MM holds, a firm is financially unconstrained and solves its value maximization problem. We summarize the first-best solution for the value-maximizing firm. First, consider the firm's value in the mature phase.

### 3.1 The mature phase

Let  $P^*(W, Y)$  denote the firm's value in the mature phase in the MM world. With perfect capital markets, the firm's value is simply given by the sum of its cash  $W$  and the value of its asset in place:

$$P^*(W, Y) = Q^*(Y) + W. \quad (6)$$

Here,  $Q^*(Y)$  is the value of the firm's asset in place, which equals the present discounted value of its future operating profits  $Y_t - Z$ . Importantly, the calculation accounts for the firm's optimal exercising of its abandonment option. For sufficiently low values of  $Y$ , i.e.  $Y \leq Y_a^*$  where  $Y_a^*$  is the optimal abandonment hurdle to be determined soon, the asset is abandoned and hence  $Q^*(Y) = 0$ .

For the more interesting case where  $Y \geq Y_a^*$ , we may write  $Q^*(Y)$  as the solution of the following ODE:

$$rQ(Y) = Y - Z + \mu Y Q'(Y) + \frac{\sigma^2 Y^2}{2} Q''(Y), \quad Y > Y_a^*, \quad (7)$$

with the standard value-matching and smooth-pasting conditions:

$$Q(Y_a^*) = 0, \quad (8)$$

$$Q'(Y_a^*) = 0. \quad (9)$$

Importantly, the abandonment hurdle  $Y_a^*$  is endogenous and part of the model solution. The value  $Q^*(Y)$  admits the following unique closed-form solution:

$$Q^*(Y) = \left( \frac{Y}{r - \mu} - \frac{Z}{r} \right) + \left( \frac{Y}{Y_a^*} \right)^\gamma \left( \frac{Z}{r} - \frac{Y_a^*}{r - \mu} \right), \quad \text{for } Y \geq Y_a^*. \quad (10)$$

The first term in (10) is the present discounted value of its operating profits if the firm were to remain in operation forever (which would be suboptimal for sufficiently low  $Y$ ). The second term gives the additional value created if the firm were to optimally exercise its abandonment option. Without capital market frictions, the firm optimally operates its physical asset if and only if  $Y \geq Y_a^*$ , where

$$Y_a^* = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{r} Z, \quad (11)$$

and the constant  $\gamma$  is given by

$$\gamma = \frac{1}{\sigma^2} \left[ - \left( \mu - \frac{\sigma^2}{2} \right) - \sqrt{\left( \mu - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] < 0. \quad (12)$$

In the mature phase, the firm's value is convex in earnings  $Y$  due to its abandonment option. Due to the option value of keeping the asset as a going concern, at the optimal abandonment hurdle  $Y_a^*$ , the present discounted value of future revenues is less than the perpetual value of the operating cost  $Z/r$ , in that

$$\frac{Y_a^*}{r - \mu} = \frac{\gamma}{\gamma - 1} \frac{Z}{r} < \frac{Z}{r}, \quad (13)$$

as  $\gamma < 0$ . Next, we turn to the firm's value in the start-up phase.

### 3.2 The start-up phase

We denote the *first-best* value of a financially unconstrained firm by  $G^*(W, Y)$ . As for the firm's value  $P^*(W, Y)$  in the mature phase, the first-best value  $G^*(W, Y)$  takes the following simple additive form:

$$G^*(W, Y) = H^*(Y) + W. \quad (14)$$

Here,  $H^*(Y)$  is the value of the firm's growth option, which includes the present discounted value of its future operating profits  $Y_t - Z$ , the value of optimal growth option exercising, and the value of abandonment. Intuitively, with perfect capital markets, again, firm value is given by the sum of its cash holding  $W$  and the value of its growth option,  $H^*(Y)$ , which can be valued independently from its liquidity holding.

For sufficiently high values of  $Y$ , i.e.  $Y \geq Y_i^*$  where  $Y_i^*$  is the optimal investment hurdle to be determined, the growth option is immediately exercised and  $H^*(Y) = Q^*(Y) - I$ , where  $Q^*(Y)$  is the value of assets in place in the mature phase, given in (10). For the more interesting case where  $Y < Y_i^*$ , we may write  $H^*(Y)$  as the solution of the following ODE:

$$rH(Y) = \mu Y H'(Y) + \frac{\sigma^2 Y^2}{2} H''(Y), \quad (15)$$

subject to the value-matching and smooth-pasting boundary conditions:

$$H(Y_i^*) = Q^*(Y_i^*) - I, \quad (16)$$

$$H'(Y_i^*) = Q^{*'}(Y_i^*). \quad (17)$$

Additionally, the growth option is worthless at the origin  $Y = 0$  as it is an absorbing state for a GBM process, i.e.  $H(0) = 0$ .

The optimal investment hurdle  $Y_i^*$  is the solution to the following equation:

$$(\beta - \gamma) \left( \frac{Y_i^*}{Y_a^*} \right)^\gamma \left( \frac{Z}{r} - \frac{Y_a^*}{r - \mu} \right) + (\beta - 1) \frac{Y_i^*}{r - \mu} - \beta \left( \frac{Z + rI}{r} \right) = 0. \quad (18)$$

The option value  $H^*(Y)$  has the following closed-form solution:

$$H^*(Y) = \left( \frac{Y}{Y_i^*} \right)^\beta (Q^*(Y_i^*) - I), \quad \text{for } Y \leq Y_i^*, \quad (19)$$

where  $\beta$  is a constant given by

$$\beta = \frac{1}{\sigma^2} \left[ - \left( \mu - \frac{\sigma^2}{2} \right) + \sqrt{\left( \mu - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] > 1. \quad (20)$$

Again, here the firm's value is convex in  $Y$ . Next, we turn to the analysis for a financially constrained firm.



## 4 Abandonment and Financing in the Mature Phase

In the mature phase, the firm manages its asset in place and has a liquidity hoard  $W$ . With external financing costs, liquidity hoard  $W$  influences its decision and its valuation. First we note that the firm will never voluntarily issue external equity provided that it has liquidity to keep the firm solvent, as the firm is better off by postponing its equity issue. Intuitively, the firm's financing term does not change and delaying equity issue saves the forgone interest income on the financing cost. Therefore, the firm can be in one of the three regions depending on its liquidity  $W$  and earnings  $Y$ :

1. the “financially unconstrained” region where more  $W$  does not influence the firm's decision in any way and the firm behaves in the same way as it does in an MM world;
2. the “interior” financially constrained region where the firm does not raise any external financing nor pays anything out but simply hoards and accumulates its liquidity  $W$  and continues its operation;
3. the “equity issuance/liquidation” region where the firm runs out of its liquidity capacity and moreover will be insolvent without external financing.

Denote the firm's value in the mature phase by  $P(W, Y)$ . We may write it as

$$P(W_t, Y_t) = \max_{\tau_L, dC \geq 0} \mathbb{E}_t \left[ \int_t^{\tau_U \wedge \tau_L} e^{-r(s-t)} (-\mathcal{I}_{dC_s > 0} \Phi(dC_s)) + e^{-r(\tau_U - t)} P^*(W_{\tau_U}, Y_{\tau_U}) \mathcal{I}_{\tau_U < \tau_L} \right],$$

where  $P^*(W_t, Y_t)$  is the first-best firm value for a financially unconstrained firm given by (6),  $\Phi(dC)$  is the cost of issuing external financing  $dC$ , and  $\mathcal{I}_{dC_s > 0}$  is an indicator function which takes the value of one when  $dC_s > 0$  and zero otherwise. Note that we have two stopping times:  $\tau_U$  is the stopping time that the firm accumulates sufficient liquidity such that it permanently becomes unconstrained and hence attains the first-best firm value  $P^*(W_t, Y_t)$ , and  $\tau_L$  is the endogenous stochastic liquidation time chosen by the firm.  $\mathcal{I}_{\tau_U < \tau_L}$  is an indicator function which takes the value of one when  $\tau_U < \tau_L$  and zero otherwise.

If liquidation is suboptimal, the firm must raise costly external financing to be able to continue operating the project should it run out of cash. That is, at any time  $s$  when the firm incurs operating losses,  $Y_s < Z$ , and also is out of cash,  $W_s = 0$ , it has to raise funds  $dC_s$  at least sufficient to cover operating losses, if it were to continue operations.

### 4.1 The financially unconstrained region

Unlike in a static setting, a firm is financially unconstrained in a dynamic setting if and only if it faces no financial constraint with probability one at the current and all future times. There are two ways that the firm can be financially unconstrained: (1) it internally generates sufficient liquidity at all times; or (2) the firm already has a sufficient liquidity hoard:  $W \geq \Lambda$ , where  $\Lambda$  denotes the lowest level of liquidity needed for a mature firm to be permanently financially unconstrained. We will provide an explicit formula for  $\Lambda$ .

*Type-1 financially unconstrained firm:  $Y \rightarrow \infty$ .*

If the firm's internally generated cash flow  $Y$  is very high, the firm will be fully liquid even without any liquidity hoard  $W$ , as internally generated revenue  $Y$  is fully sufficient to cover its flow operate cost  $Z$  per period without ever having to raise external funds. In the limit where  $Y \rightarrow \infty$ , the following boundary condition must hold:

$$\lim_{Y \rightarrow \infty} P(W, Y) = \lim_{Y \rightarrow \infty} P^*(W, Y), \quad (21)$$

where  $P^*(W, Y)$  is the value for a financially unconstrained firm and is given by (6).

*Type-2 financially unconstrained firm:  $W \geq \Lambda$ .*

The firm is able to implement the first-best abandonment option policy and avoid costly external financing permanently with probability one provided that the firm has sufficiently high liquidity hoarding. The question is how high the firm's liquidity hoard  $W$  has to be for a firm to be permanently unconstrained. As long as the firm does not abandon its asset in place prematurely and involuntarily, that is, when  $Y > Y_a^*$ , the firm achieves its first-best policy and thus it is financially unconstrained. That is, as long as the corporate saving rate  $(rW + Y - Z)$  is weakly positive at  $Y_a^*$ , then the firm will never be forced to liquidate the firm sub-optimally. That is, we need  $rW + Y_a^* - Z \geq 0$ , which implies

$$W \geq \frac{Z - Y_a^*}{r} = \Lambda. \quad (22)$$

Using the explicit formula (22) for the abandonment hurdle  $Y_a^*$ , we write  $\Lambda$  as

$$\Lambda = \frac{r - \gamma\mu}{r^2(1 - \gamma)}Z. \quad (23)$$

In summary, as long as condition (22) holds, the firm is permanently financially unconstrained, and hence the firm's value  $P(W, Y)$  equals the first-best firm's value:

$$P(W, Y) = P^*(W, Y), \quad \text{for } W \geq \Lambda. \quad (24)$$

Note that the firm can finance its efficient continuation entirely out of its cash hoard in the first-best continuation region,  $Y \geq Y_a^*$ . And for  $Y < Y_a^*$ , the firm voluntarily and efficiently abandons its asset and distributes  $W$  to shareholders.

In macro savings literature, e.g. Aiyagari (1994), a core concept is the *natural borrowing limit*, which refers to the maximal amount of risk-free credit that a consumer can tap with no probability of ever defaulting. Hence, the consumer can borrow at the risk-free rate up to that limit, but any additional amount of borrowing will give rise to default risk. Here is the analogy. For a firm,  $\Lambda$  given in (23) is the the minimum of the liquidity hoard that it needs in order to implement its first-best abandonment strategy. Any liquidity hoard lower than  $\Lambda$  may induce underinvestment via inefficient liquidation in the future with positive probability.

Next, we turn to the "interior" region where the firm hoards liquidity and operates its asset in place. In this region, the firm is financially constrained but does not raise any external financing as the firm strictly prefers deferring equity issuance decision to the future.

## 4.2 The interior liquidity hoarding region

Using the standard principle of optimality, we characterize the firm's value  $P(W, Y)$  as the solution to the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rP(W, Y) = (rW + Y - Z)P_W(W, Y) + \mu Y P_Y(W, Y) + \frac{\sigma^2 Y^2 P_{YY}(W, Y)}{2}, \quad (25)$$

subject to various boundary conditions to be discussed later. The first term on the right side of (25), given by the product of the firm's marginal value of cash  $P_W(W, Y)$  and the firm's saving rate  $(rW + Y - Z)$ , represents the effect of the firm's savings on its value. The second term (the  $P_Y$  term) represents the marginal effect of expected earnings change  $\mu Y$  on firm value, and the last term (the  $P_{YY}$  term) encapsulates the effects of the volatility of changes in earnings  $Y$  on firm value. Intuitively, the expected change of firm value  $P(W, Y)$ , given by the right side of (25), equals  $rP(W, Y)$ , as the firm's expected return is  $r$ .

With financial frictions, liquidity generally is more valuable than its pure monetary value. Typically, the firm's decision of whether to abandon the project or not is influenced by its financial considerations and the prospect of having to incur external financing costs in the future. All else equal, the costs of external financing ought to be an additional inducement to abandon a project yielding low revenues. Therefore, one would expect that the prospect of having to incur external financing costs would lower the firm's valuation for its asset in place and result in a higher abandonment hurdle.

Now consider the situation where a financially constrained firm is just indifferent between abandoning the firm or not. At the moment of indifference, firm value must be continuous,

$$P(W, \underline{Y}) = W, \quad (26)$$

which states the the firm's value equals to its liquidity hoard  $W$  at the moment of abandoning its asset. Equation(26) implicitly defines the abandonment hurdle  $\underline{Y}(W)$ . Moreover, since the abandonment decision is optimally made, the marginal values along both  $Y$  and  $W$  dimensions shall also be matched before and after the abandonment of the asset in place,

$$P_Y(W, \underline{Y}(W)) = 0. \quad (27)$$

The smooth-pasting condition (27) states that at the optimal abandonment hurdle  $\underline{Y}(W)$ ,  $P_Y$  equals zero.<sup>2</sup> Next, we turn to the equity issuance/liquidation region.

## 4.3 The equity issuance/liquidation region

As Bolton, Chen, and Wang (2011) show, in a world with constant financing opportunities where financing terms do not change over time, the firm has no need to issue equity unless it absolutely has to. By delaying equity issuance, the firm saves the time value of money for the financing costs. In the mature phase, with positive liquidity hoarding  $W$ , the firm has

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<sup>2</sup>As a result, the firm's marginal value of liquidity  $P_W(W, \underline{Y}(W))$  evaluated at the optimal abandonment hurdle  $\underline{Y}(W)$  equals one.

sufficient slack to cover any operating losses over a given small time interval. Therefore, the firm never issues equity before it exhausts its cash.

When the firm runs out of its cash ( $W = 0$ ), it finds itself in one of the three regions. First, when  $Y > Z$ , the firm is solvent even without savings as its internally generated cash flow  $Y$  covers its operating cost  $Z$ , and hence the firm needs no external financing. When the firm's internally generated revenue cannot fully cover its operating cost ( $Y < Z$ ) and with no savings ( $W = 0$ ), the firm has an option to either issue equity or to simply liquidate its asset in place altogether, whichever is in the interest of its shareholders. Intuitively, whether the firm issues equity or liquidates itself depends on how valuable the firm's going concern value is, i.e. how high the revenue  $Y$  is.

With a fixed equity issuance cost  $\phi_1 > 0$ , the firm optimally chooses the amount of financing  $F$  and the endogenous hurdle  $\underline{Y}(0)$  to satisfy the following value-matching boundary conditions at  $W = 0$ :

$$P(0, Y) = \begin{cases} P(F, Y) - F - \Phi(F), & \underline{Y}(0) < Y < Z, \\ 0, & Y \leq \underline{Y}(0). \end{cases} \quad (28)$$

The endogenous hurdle  $\underline{Y}(0)$  given in condition (28) provides the boundary between the equity-issuance region and the liquidation region,  $Y \leq \underline{Y}(0)$ . The first case in (28) corresponds to the equity issuance region,  $\underline{Y}(0) < Y < Z$ . Let  $F_a(Y)$  denote the external financing amount  $F$  as a function of earning  $Y$  (recall that financing only occurs when  $W = 0$ .) And the second case in (28) corresponds to the liquidation region,  $Y \leq \underline{Y}(0)$ . By the firm's revealed preference, the dividing boundary  $\underline{Y}(0)$  shall be higher than the first-best abandonment hurdle  $Y_a^*$ , a form of underinvestment.

Should the firm seek to raise new funds, its optimal external financing amount  $F$  is given by the first-order condition (FOC)<sup>3</sup> for the case with  $\phi_0 > 0$ :

$$P_W(F, Y) = 1 + \Phi'(F) = 1 + \phi_1, \quad \underline{Y}(0) < Y < Z. \quad (29)$$

Intuitively, conditional on the firm's issuance decision at  $W = 0$ , the marginal value of cash  $P_W(F, Y)$  equals the marginal cost of financing  $1 + \Phi'(F)$ . Note that the firm's marginal value of cash  $P_W(W, Y)$  depends on its revenue  $Y$ , and  $P_W(W, Y)$  is greater than 1 at the moment of financing.

**Summary.** In a dynamic environment, the condition for a firm to be financially unconstrained is much tighter than in a static setting. The reason is as follows. For a firm to be financially unconstrained in a dynamic setting, the firm cannot have demand for funds at any moment. That is, with probability one, the firm has no demand for funding. Only under this condition, can the firm be assured that its marginal value of cash is one. In our model, the firm can be financially unconstrained in one of the two ways; it either internally generates enough amount of funds or it has sufficient liquidity to cover all the potential needs.

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<sup>3</sup>We will verify the second-order condition (SOC) to ensure that the FOC solution yields the maximal value.

We next turn to the firm's problem in the start-up phase. Obviously, a rational forward-looking firm fully anticipates its financial constraints in the mature phase and acts accordingly in the start-up phase.

## 5 Investment and Financing in the Start-up Phase

In the start-up phase, the firm maximizes its present value by solving the optimal investment timing problem. Let  $G(W_t, Y_t)$  denote this value. Specifically, the firm chooses the optimal investment timing  $\tau_i$  to maximize the value of the growth option by solving

$$G(W_t, Y_t) = \max_{\tau_i, F \geq 0} \mathbb{E}_t \left[ e^{-r(\tau_i - t)} (P(W_{\tau_i} + F - I, Y_{\tau_i}) - (F + \Phi(F)) \mathcal{I}_{F > 0}) \right],$$

where  $\tau_i$  is the *endogenous* investment timing, and  $\mathcal{I}_{F > 0}$  is an indicator function which takes the value of one when  $F > 0$  and zero otherwise. Recall that  $P(W, Y)$  is the firm's value in the mature phase. To be able to invest, the firm must have total available funds  $W + F$  that cover at least the investment outlay  $I$ , i.e.,  $W + F \geq I$ .

When choosing its optimal investment timing, the firm incorporates both the one-time lumpy investment cost  $I$  and also its future operating (flow) cost  $Z$ . Before analyzing the effect of financial constraints on the firm's investment option exercising and financing decisions, we first reason how much liquidity the firm needs in order to be financially unconstrained.

### 5.1 The financially unconstrained region: $W \geq I + \Lambda$

For a firm to be dynamically financially unconstrained, it should have the first-best investment and abandonment decisions under all circumstances. Intuitively, with liquidity hoard greater than  $\Lambda + I$ , then with probability one, it can cover both its investment cost  $I$  and its future liquidity shortfall to continue an efficient operation of its asset with liquidity amount  $\Lambda$ . Therefore, the firm in its start-up phase is financially unconstrained if and only if

$$W \geq I + \Lambda. \tag{30}$$

In summary, as long as (30) holds, the firm is permanently financially unconstrained, and hence in the startup phase, its value  $G(W, Y)$  achieves the first-best value given by

$$G(W, Y) = G^*(W, Y) = H^*(Y) + W, \quad \text{for } W \geq I + \Lambda, \tag{31}$$

where  $H^*(Y)$  is given by (19) and the first-best investment hurdle  $Y_i^*$  is given by (18). Recall that  $H^*(Y)$  and  $Y_i$  are independent of liquidity  $W$ . Note that in the mature phase, the firm can also finance its efficient continuation entirely out of its cash hoard with an optimal abandons its asset and distributes  $W$  to shareholders.

When  $W < I + \Lambda$ , the firm is financially constrained. There are two sub-cases:

- When  $I \leq W < I + \Lambda$ , the firm has a sufficient liquidity hoard to fund the investment outlay  $I$  entirely out of its internal funds, but may not have sufficient funds to avoid the liquidity shortage in the mature phase and hence equity issuance or involuntary liquidation may occur with positive probability.

- When  $W < I$ , the firm cannot cover its investment cost  $I$ , and hence the firm may require external financing to cover both the investment cost and the liquidity need in the mature phase.

Note that a financially constrained firm has a option value of building up financial slack internally. The tradeoff between internal and external financing, the timing decision, and the consideration of future liquidity needs in the mature phase make the financially constrained firm's decision a complex but very important one.

## 5.2 The Medium Cash-holding Region: $I \leq W < I + \Lambda$

Consider now the situation of a firm with moderate financial slack. This firm has sufficient internal funds  $W$  to cover the investment cost  $I$ , but not quite enough cash to ensure that it will never run out of internal funds in the mature phase:  $I \leq W < I + \Lambda$ . In the forward-looking sense, the firm is still financially constrained and the marginal value of cash is greater than one. For such a firm it is optimal not to raise any external funds when it *chooses* to exercise its growth option, so that  $F = 0$ . Note that the firm may not choose to exercise the growth option. Importantly, the firm realizes that exercising the investment option drains its cash holding by  $I$  and hence the firm may be led to raise external funds in the future to cover operating losses in the mature phase. Therefore, the firm is still financially constrained, as it may have potential liquidity demand in the mature phase.

In the waiting region, the firm's value  $G(W, Y)$  solves the following HJB equation:

$$rG(W, Y) = rWG_W(W, Y) + \mu Y G_Y(W, Y) + \frac{\sigma^2 Y^2}{2} G_{YY}(W, Y), \quad (32)$$

subject to various boundary conditions to be discussed later. Note that the first term on the right side of (32) reflects the firm's savings effect on firm value. The remaining two terms are the standard earnings drift and volatility effects on the option value.

As in the standard real options literature, at the endogenously chosen moment of investment, firm value  $G(W, Y)$  is continuous and hence

$$G(W, \bar{Y}) = P(W - I, \bar{Y}). \quad (33)$$

The value-matching condition (33) characterizes the investment hurdle as an implicit function of liquidity  $W$ ,  $\bar{Y}(W)$ . In this region, the investment cost  $I$  is entirely financed out of internal funds, and hence liquidity  $W$  decreases by  $I$ , as seen on the right side of (33). Additionally, because the investment hurdle  $\bar{Y}(W)$  is optimally chosen, we have the following smooth-pasting condition along the earning  $Y$  dimension:

$$G_Y(W, \bar{Y}(W)) = P_Y(W - I, \bar{Y}(W)). \quad (34)$$

Finally, when the absorbing state  $Y = 0$  is reached, there is no investment opportunity and the only valuable asset of the firm is its cash  $W$ , and hence

$$G(W, 0) = W. \quad (35)$$

Next, we turn to the low cash-holding region where investment cannot be financially solely with internal funds,  $W < I$ .

### 5.3 The Low Cash-holding Region, $0 \leq W < I$

In the region where internal funds are insufficient to cover the investment cost  $I$ , the firm has to raise external financing should it decide to invest. Intuitively, no matter how large its current earning  $Y$  is, the firm has to access external capital markets if choosing to *immediately* invest, as the investment cost  $I$  is lumpy while the earning  $Y$  is a flow. At the moment of investing, the firm's value must be continuous, i.e.,

$$G(W, \bar{Y}) = P(W + F - I, \bar{Y}) - F - \Phi(F). \quad (36)$$

The right side of the value-matching condition (36) gives the firm's value after it issues net amount  $F$  and incurs a cost  $\Phi(F)$ . The left side of (36) is the firm's value before investing. Note that the post-financing/investment liquidity is  $W + F - I$ .

Of course, it is quite plausible that the firm may want to wait. In this waiting region, the firm's value  $G(W, Y)$  also solves the HJB equation (32) for the same argument as the one used in the previous subsection (in the medium cash-holding region). In addition to the investment hurdle  $\bar{Y}(W)$ , the firm also needs to choose the net equity issue amount  $F$  to at least cover the needed financing for investment  $I - W$ .

We denote by  $F_g(W)$  the amount of external financing by the firm as a function of  $W$  in the region  $0 \leq W < I$ . The minimal amount of issuance required so that the post-issuance liquidity is non-negative is  $I - W$ . Thus, under optimal external financing the following inequalities must hold:

$$P_W(W + F_g(W) - I, \bar{Y}(W)) \leq 1 + \Phi'(F_g(W)) \text{ and } F_g(W) \geq I - W. \quad (37)$$

That is, the firm will issue equity such that the marginal value of liquidity is weakly lower than the marginal cost of issuance. We write the optimality conditions in this way because it is possible that the constraint  $F_g(W) \geq I - W$  may bind, in which case the firm chooses to rely solely on its ability to generate sufficient liquidity from operating earnings after it has invested in the productive asset.

The firm's optimality implies that the marginal value of earning  $Y$  is continuous before and after the investment option is exercised. Therefore, we have the following two smooth-pasting conditions:

$$G_Y(W, \bar{Y}(W)) = P_Y(W + F_g(W) - I, \bar{Y}(W)). \quad (38)$$

Finally,  $G(W, 0) = W$  as  $Y = 0$  is an absorbing state.

## 6 Analysis

As is standard in the literature, we set the risk-free interest rate  $r = 5\%$ , the expected earnings growth rate  $\mu = 0$ , and the earnings growth volatility  $\sigma = 15\%$ . The investment cost is set at  $I = 2$  and the operating cost is  $Z = 1$ . When applicable, the parameter values are annualized.

The first-best liquidation hurdle is  $Y_a^* = 0.625$  which implies that a financially unconstrained firm will continue as a going concern even when it incurs a loss of  $Z - Y_a^* = 0.375$ , 37.5% of the (flow) operating cost  $Z = 1$ . This indicates a significant option value for a firm to continue as a going concern under MM. In the startup phase, the firm exercises its growth option when its earnings  $Y$  reaches the first-best investment hurdle  $Y_i^* = 1.544$ . At the moment of exercising, the value of the assets in place (including the option value of abandonment) is  $Q^*(Y_i^*) = 12.54$ . As the investment cost  $I = 2$ , at the moment of investment, the firm's value (netting the investment cost) is  $H^*(Y_i^*) = 10.54$ .

With sufficiently high cash holding  $W$ , the firm is financially unconstrained at all times with probability one. In our example, the minimal amount of liquidity needed for a firm in the mature phase to be permanently financially unconstrained is

$$\Lambda = \frac{r - \gamma\mu}{r^2(1 - \gamma)}Z = 7.50, \quad (39)$$

which is 7.5 times the operating cost  $Z = 1$ . In the startup phase, the minimal amount of liquidity needed for the firm to be permanently financially unconstrained is thus  $\Lambda + I = 9.50$  which covers both the investment cost  $I$  and the liquidity needs in the mature phase.

For the external financing cost, we choose the marginal issuance cost  $\phi_1 = 0.01$  motivated by the empirical analysis in Altinkilic and Hansen (2000). The fixed equity issuance cost induces lumpy issuance, which is empirically important. We thus focus on the parameter  $\phi_0$  in our comparative static analysis. For our baseline case, we choose  $\phi_0 = 0.4$  which implies that the fixed equity issuance cost is about 4% of  $H^*(Y_i^*) = 10.54$ , the first-best (net) firm value at the moment of investment and also this value is broadly in line with the empirical estimate reported in Altinkilic and Hansen (2000). To highlight the impact of the fixed financing cost  $\phi_0$ , we thus consider three values:  $\phi_0 = 0.01, 0.4, 2$ . By varying  $\phi_0$  we see how the financing optionality interacts with the real optionality.

## 6.1 The Mature Phase

We define the firm's *enterprise value* as its total value in excess of cash,

$$Q(W, Y) = P(W, Y) - W. \quad (40)$$

Because cash is valuable beyond its face value for a financially constrained firm, the enterprise value also depends on  $W$ . Under the MM condition, the enterprise value is independent of the firm's cash holding, and we have  $Q^*(Y) = P^*(W, Y) - W$ , as given by (6).

**The Liquidation decision.** Figure 1 plots the optimal liquidation hurdle  $\underline{Y}(W)$  for a financially constrained firm. First note that the firm becomes permanently financially unconstrained when its cash holding reaches  $\Lambda = 7.50$ , which is 7.5 times the (annual) operating cost  $Z = 1$ . For a financially unconstrained firm, it is optimal to liquidate its asset when its earning  $Y$  falls below the first-best liquidation hurdle  $Y_a^* = 0.625$ . And importantly, the firm will never be forced into sub-optimally abandoning its asset due to the shortage of its



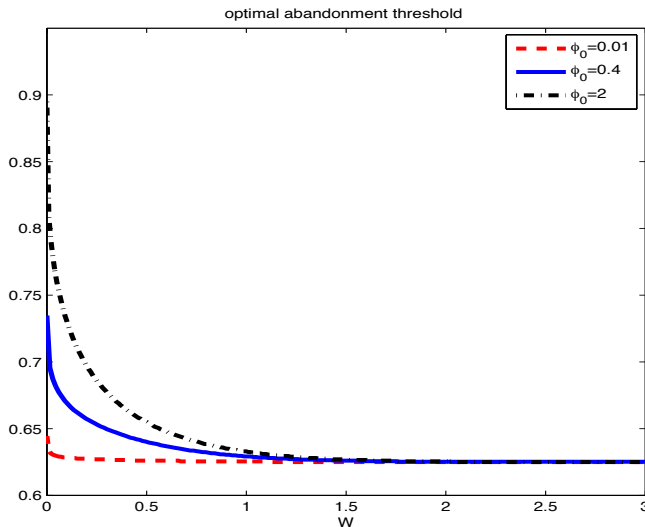


Figure 1: **The optimal liquidation hurdle  $\underline{Y}(W)$  for a financially constrained firm in the mature phase.** The endogenous liquidation hurdle  $\underline{Y}(W)$  is monotonically decreasing with liquidity holding  $W$  and approaches the first-best level  $Y_a^* = 0.625$  independent of the financing cost  $\phi_0$ . For a given value of  $W$ , the larger the fixed equity issuance cost  $\phi_0$ , the higher the liquidation hurdle  $\underline{Y}(W)$  indicating a higher degree of under-investment. At  $W = 0$ ,  $\underline{Y}(0)$  equals 0.645, 0.735, 0.895 for  $\phi_0 = 0.01, 0.4, 2$ , respectively.

liquidity  $W$  as long as  $W \geq \Lambda = 7.50$ . Quantitatively, Figure 1 shows that the firm with liquidity  $W$  larger than 3 is effectively dynamically financially unconstrained.

When the firm exhausts its internal funds  $W_\tau = 0$  and its cash flow is larger than its operating cost  $Z$  (i.e.,  $Y > Z = 1$ ), the firm remains solvent purely relying on its internally generated cash flow. However, when its earning  $Y$  falls below its operating cost (i.e.,  $Y < 1$ ), it has to either raise external funds or will be liquidated otherwise. In our baseline case where  $\phi_0 = 0.4$ , if a constrained firm's earning  $Y$  lies in the region  $1 > Y > \underline{Y}(0) = 0.735$ , it is optimal to issue equity keeping the firm alive. Only when its earnings  $Y$  falls below 0.735, the firm will abandon its asset rather than issue equity to finance its liquidity shortfall. Recall that the first-best liquidation boundary is  $Y_a^* = 0.625$ . Hence, a firm with  $W = 0$  will be inefficiently liquidated due to lack of internal funds in the region  $0.625 < Y \leq 0.735$ .

Figure 1 also illustrates that the liquidation hurdle  $\underline{Y}(W)$  decreases with the firm's liquidity  $W$  in the region  $[0, \Lambda] = [0, 7.50]$  for all three levels of  $\phi_0$ . Hence, inefficiently liquidation occurs in the region  $0 \leq W < \Lambda$ , as  $P_W > 1$  in this region and liquidity is valuable. For example, when  $\phi_0 = 2$ , the abandonment hurdle decreases from  $\underline{Y}(0) = 0.895$  to  $Y_a^* = 0.625$  as  $W$  increases from the origin to  $\Lambda = 7.50$ . Intuitively, the higher the liquidity holding  $W$ , the less inefficient the firm's liquidation decision. For the case with a small fixed equity issuance cost,  $\phi_0 = 0.01$ , the impact of financial constraints is negligible; the cashless firm will only be abandoned inefficiently if its earning  $Y$  falls inside the tight region  $0.625 < Y \leq \underline{Y}(0) = 0.645$ . How important is the impact of financing costs on liquidation?

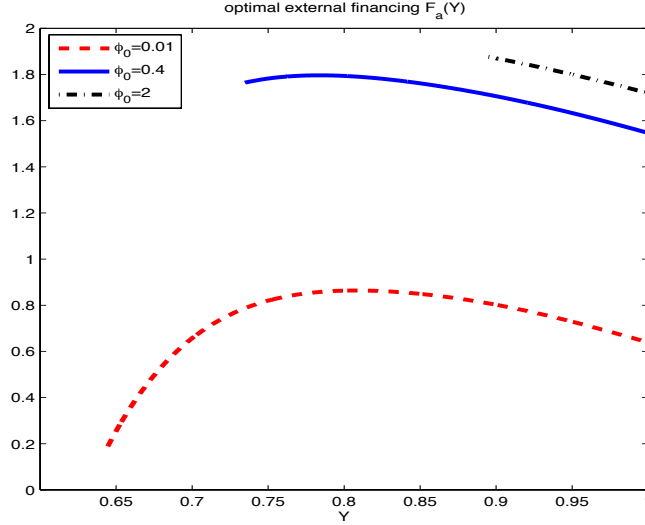


Figure 2: **The optimal external financing  $F_a(Y)$  for a financially constrained firm in the mature phase.** Firms will choose to raise external funds only when it runs out of its liquidity and when its internally generated cash flow cannot cover the operating cost  $Z$  but is sufficiently high (i.e., only when  $W = 0$  and  $\underline{Y}(0) < Y < Z = 1$ ), where  $\underline{Y}(0)$  is the optimal abandonment hurdle for a financially constrained firm. For  $\phi_0 = 0.01, 0.4, 2$ , we have shown that  $\underline{Y}(0) = 0.645, 0.735, 0.895$ , respectively. Interestingly, the firm's net financing amount  $F_a(Y)$  is non-monotonic in its earning  $Y$  over the region  $(\underline{Y}(0), 1)$ . For example, for the case with  $\phi_0 = 0.4$ , the external financing  $F_a(Y)$  first increases in the region  $Y \in (0.735, 0.80)$ , peaks at  $Y = 0.80$  with a value of  $F_a(0.80) = 1.80$ , and then decreases with  $Y$  in the region  $Y \in (0.80, 1)$ .

At the origin  $W = 0$ , as we increase the fixed issuance cost  $\phi_0$  from 0.01 to 0.4 and then from 0.4 to 2, the abandonment hurdle  $\underline{Y}(0)$  increases from 0.645 to 0.735 and then from 0.735 to 0.895, respectively. The implied real inefficiencies are significant. Finally, we note that quantitatively the effect of financial constraints essentially disappears as the firm's liquidity hoard  $W$  reaches 1.6, even when the fixed equity issuance cost is relatively high,  $\phi_0 = 2$ .

**The Equity Issuance Decision.** The firm will consider the possibility of issuing equity when it runs out of cash ( $W = 0$ ) and its earning cannot cover its operating cost ( $Y < Z$ ). Intuitively, it is always preferable for the firm to postpone raising external funds whenever feasible. Additionally, the firm will not issue equity if its earning falls below its first-best abandonment hurdle  $Y_a^*$ , as it must be optimal for a financially constrained firm to abandon its assets in place if it is optimal for a financially unconstrained firm to do so. Hence, in Figure 2, we only need to plot the optimal equity issuance amount  $F_a(Y)$  as a function of its earning  $Y$  in the region  $[Y_a^*, Z] = [0.625, 1]$  for the three cases,  $\phi_0 = 0.01, 0.4, 2$ .

Importantly, the amount of equity financing  $F_a(Y)$  may be non-monotonic in  $Y$ . For the baseline case with  $\phi_0 = 0.4$ ,  $F_a(Y)$  first increases with  $Y$  in the region  $\underline{Y}(0) < Y < 0.80$

and then decreases with  $Y$  in the region  $0.80 < Y < 1$ . The net issuance amount  $F$  peaks at  $Y = 0.80$  with a value of  $F_a(0.80) = 1.80$ . In the region  $0.735 \leq Y < 0.80$ , the firm's future prospects are not sufficiently encouraging for it to raise much external funding as the liquidation threat (in the future) is not low. This is the dominant consideration in this region, so that when  $Y$  increases, it is marginally worth raising more funds (paying the marginal cost  $\phi_1 = 0.01$ ) given that the firm's survival likelihood is improving. In the region where  $Y \in (0.80, 1)$  on the other hand, the dominant consideration when  $Y$  increases is the greater likelihood that the firm will be able to generate funds internally via its earning  $Y$  and interest income  $rW$  in the near future so that the firm does not need much external funds. As a result the firm optimally chooses to rely less on current external financing in the expectation of larger future internally generated funds. Intuitively, the firm's expectation about its own future ability to generate internal funds (e.g., from operations and interest incomes) significantly influences the firm's current financing policy.

Additionally, conditional on choosing to raise funds, the firm raises more if the the fixed costs of external funding  $\phi_0$  are higher. For example, at  $Y = 0.9$ , as we increase the fixed issuance cost  $\phi_0$  from 0.01 to 0.4 and then from 0.4 to 2, the firm's external financing  $F_a(0.9)$  increases from 0.81 to 1.71, and then from 1.71 to 1.88. Intuitively, a firm that faces a larger fixed issuance cost  $\phi_0$  has a stronger incentive to issue more and hence capitalize on the fixed cost  $\phi_0$  as the cost of going back to the capital markets is greater in the future *ceteris paribus*. Importantly, this prediction is the opposite to that based on the intuition from static models such as Froot, Scharfstein, and Stein (1993) and Kaplan and Zingales (1997). In these static models, the higher the financing cost, the more constrained the firm, and the lower the amount of equity financing.

Figure 3 plots the firm's enterprise value  $Q(W, Y)$  and its *marginal enterprise value of cash*  $Q_W(W, Y)$  against liquidity  $W$  for two different levels of earning,  $Y = 0.65$  and  $Y = 1.5$ , and for three fixed equity issuance costs,  $\phi_0 = 0.01, 0.4, 2$ . Intuitively, the higher the financing cost  $\phi_0$ , the lower the firm's enterprise value  $Q(W, Y)$ . Also note that the *net marginal value of cash*  $Q_W(W, Y)$  is always positive implying that the firm is financially constrained and hence liquidity is valuable in the constrained region, i.e.,  $W < \Lambda = 7.50$ .

**The firm's value and the marginal value of liquidity.** A central observation emerging from Figure 3 is that the firm's marginal enterprise value of cash  $Q_W(W, Y)$  can vary non-monotonically with its liquidity  $W$ . Panel B (with  $Y = 0.65$ ) highlights the non-concavity of  $Q(W, Y)$  in  $W$ . Specifically,  $Q(W, Y)$  can be either concave or convex in liquidity  $W$ . First, consider the case with a small fixed equity issuance cost,  $\phi_0 = 0.01$ . Even with a low earning, e.g.,  $Y = 0.65$ , the firm will not abandon its asset as the firm's abandonment hurdle  $\underline{Y}(0) = 0.645$  and hence the firm will never liquidate. In this case, liquidity provides value by mitigating the firm's external funding needs. Hence, the firm's liquidity is valuable and firm value is concave in  $W$  with the marginal value  $Q_W(W, Y)$  monotonically decreasing from 0.280 to zero as  $W$  increases from zero to  $\Lambda = 7.50$ . However, as we increase the fixed cost  $\phi_0$  from 0.01 to 0.4, the marginal enterprise value of liquidity  $Q_W(W, Y)$  is no longer concave. Indeed,  $Q_W(W, Y)$  first equals zero for  $W \leq 0.29$  and then increases for  $0.29 < W < 0.6$  and finally decreases with  $W$  in the region with sufficiently high  $W > 0.6$ .

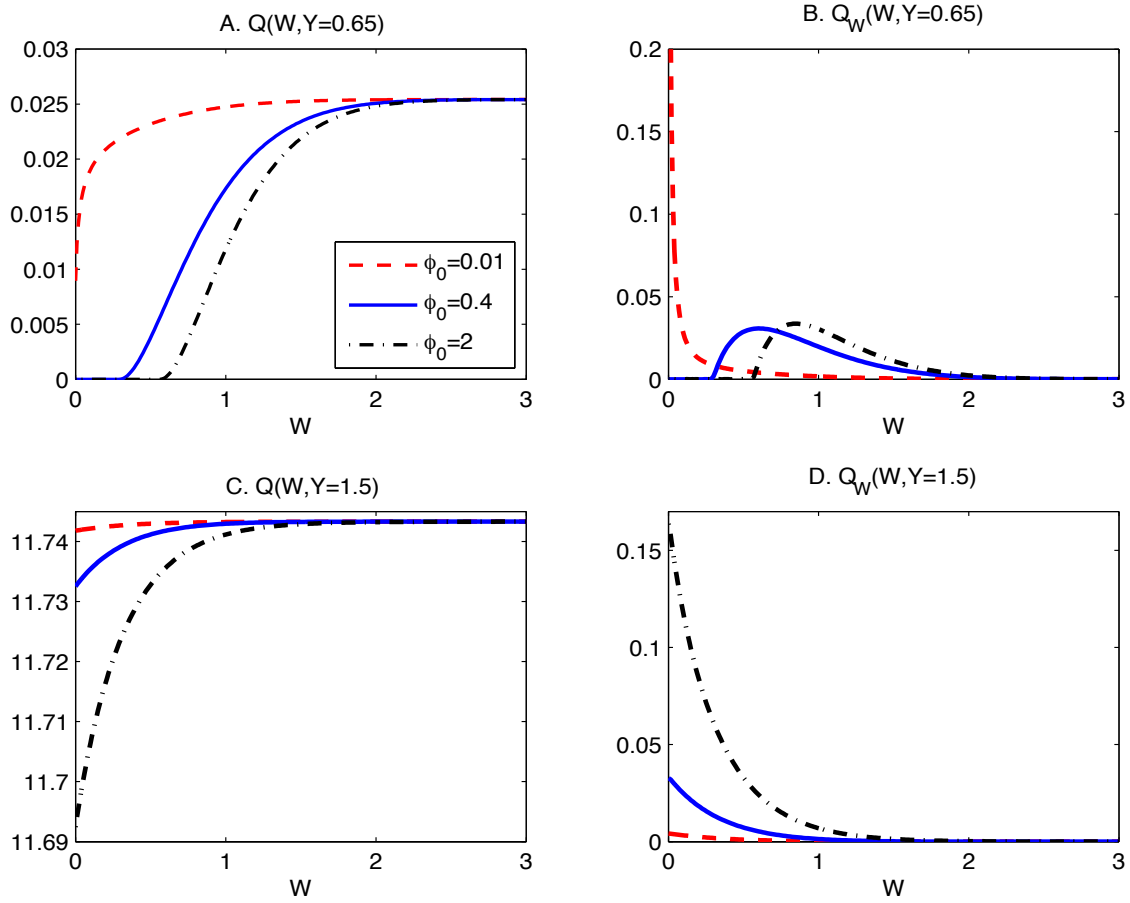


Figure 3: **Enterprise value  $Q(W, Y)$  and the marginal enterprise value of cash  $Q_W(W, Y)$  in the mature phase.** We plot  $Q(W, Y)$  and  $Q_W(W, Y)$  as functions of liquidity  $W$  for three earning levels ( $Y = 0.65, 1.5$ ) and for three values of the fixed equity issuance cost,  $\phi_0 = 0.01, 0.4, 2$ .

The intuition is as follows. A firm with a low earning  $Y = 0.65$  optimally chooses to exercise its abandonment option provided that its liquidity  $W \leq 0.29$ , as the abandonment hurdle  $\underline{Y}(0.29) = 0.65$ . Hence, in the low liquidity region  $0.29 < W < 0.6$ , increasing  $W$  lowers the firm's likelihood of tapping costly external financing, but the firm still has a low probability to survive. In this case, the firm may be endogenously risk-loving with respect to exercising its equity issuance option. This can only be the case if the firm faces strictly positive fixed costs of equity issuance ( $\phi_0 > 0$ ). And hence  $Q_W$  increases with  $W$  in this low-liquidity region implying that firm value is *locally convex* in  $W$  in this region. Finally, when  $W > 0.6$ , the firm has sufficiently high liquidity and equity issuance becomes much less likely causing the marginal enterprise value of liquidity  $Q_W$  to decrease as the firm becomes less financially constrained. We also see that with a higher  $\phi_0$  (an increase of  $\phi_0$  from 0.4 to 2) shifts  $Q_W(W, Y)$  as a function of  $W$  to the right while retaining the general non-monotonic shape. This makes sense as a firm facing a larger external equity financing cost  $\phi_0$  chooses a higher abandonment hurdle  $\underline{Y}(W)$ .

Now, we consider the case with a relative high earning, e.g.,  $Y = 1.5$ , the firm's enterprise value  $Q(W, Y)$  is now concave in  $W$ , as its earning is significantly larger than the abandonment hurdle  $\underline{Y}(W)$  and the financing option is sufficiently deeply in the money. In these cases, the firm behaves as if it were risk averse ( $P_{WW} < 0$ ) with respect to liquidity  $W$ .

## 6.2 The Start-up Phase

As for the mature phase, we again define the firm's *enterprise value* in the *start-up* phase as

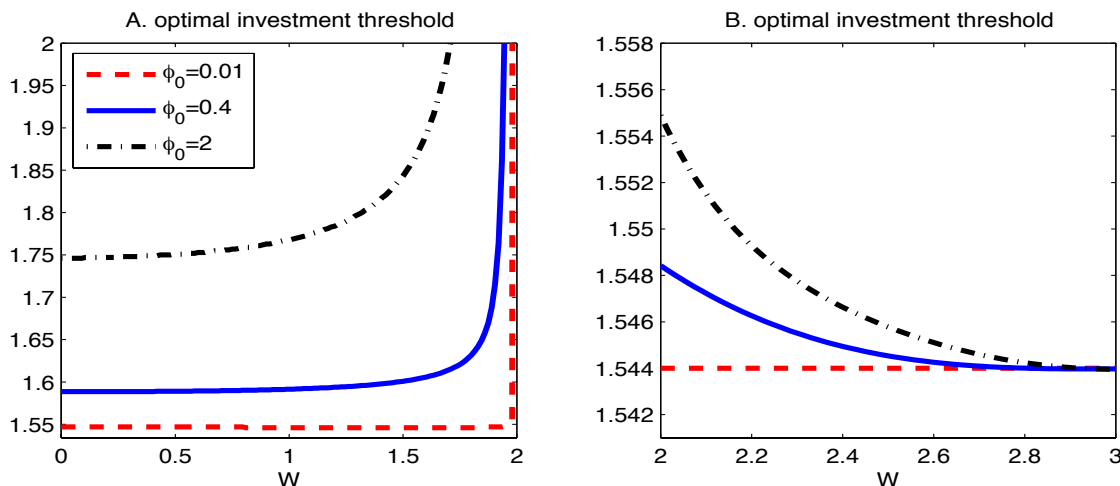
$$H(W, Y) = G(W, Y) - W. \quad (41)$$

For a financially unconstrained firm, its enterprise value is independent of its cash holding, i.e.,  $H^*(Y) = G^*(W, Y) - W$ , and the closed-form solution for  $H^*(Y)$  is given by (19). In general, a financially constrained firm's enterprise value depends on its cash holding  $W$  and is lower than the first-best value,  $H(W, Y) \leq H^*(Y)$  due to investment timing distortions.

**The Investment Decision.** Figure 4 plots a financially constrained firm's optimal investment hurdle  $\bar{Y}(W)$  for three values of the fixed cost,  $\phi_0 = 0.01, 0.4, 2$ . Recall that in the start-up phase, a firm is financially unconstrained if its liquidity holding is greater than  $I + \Lambda = 9.50$ . The firm pays the cost of  $I = 2$  to exercise the investment option if and only if the earning  $Y$  exceeds the endogenously chosen first-best time-invariant investment hurdle  $Y_i^* = 1.544$ .

First, consider the region where  $2 \leq W < 9.5$ . In this medium cash-holding region, the firm's internal funds  $W$  are sufficient to cover its investment cost  $I = 2$  and hence there will be no equity issuance decision. The higher the liquidity  $W$ , the less distorted investment timing decision. Panel B of Figure 4 shows the decreasing function of  $\bar{Y}(W)$  against liquidity  $W$ . For the case with  $\phi = 2$ , the investment hurdle decreases from  $\bar{Y}(2) = 1.555$  to the  $Y_a^* = 1.544$  as  $W$  increases from 2 to 3. Note that the quantitative effect of the financing friction in this medium cash-holding region is small.

Figure 4: **The optimal investment hurdle  $\bar{Y}(W)$  as a function of pre-investment liquidity  $W$ .**



In contrast, in the low cash-holding region  $W \in [0, I)$ , the quantitative effect of financing costs on investment is much greater. Panel A of Figure 4 plots the investment hurdle  $\bar{Y}(W)$  as a function of  $W$  in the region  $[0, 2)$ . Consider first the case with  $\phi_0 = 0.01$ . In this case,  $\bar{Y}(W)$  is non-monotonic in  $W$ ; it first decreases slightly from 1.547 to 1.545 as  $W$  increases from  $W = 0$  to  $W = 1.85$ . This is intuitive: as  $W$  increases the firm can rely on a greater and greater portion of internal funds, thus lowering its overall cost of investment, so that it is willing to take on investments with lower and lower earning  $Y$  mitigating distortions due to delayed investment timing decision. But eventually  $\bar{W}$  rises with  $W$ , and *asymptotes* to  $\infty$ , as  $W \mapsto I = 2$ . Thus, far from behaving more and more like an unconstrained firm as  $W \mapsto I$ , it behaves like a more and more constrained firm. What is the logic behind this behavior? In essence, as  $W \mapsto I$  the opportunity cost to the firm of entirely avoiding the fixed external financing cost  $\phi_0 = 2$  by waiting until it has accumulated sufficient internal funds to pay for the investment outlay  $I$  becomes smaller and smaller. It therefore takes a larger and larger earning shock  $Y$  to get the firm to invest right away and incur the cost  $\phi_0 = 2$  rather than wait a little, accumulate  $W$  at the interest rate  $r$  until  $W \geq I$ , and avoid this cost entirely.

One implication of this result is that firms facing fixed external financing costs and that have insufficient internal funds to cover their cost of investment tend to delay their capital expenditures relative to unconstrained firms. Whited (2006) estimates a model of lumpy investment by financially constrained and unconstrained firms, and finds evidence that financially constrained firms do tend to spread out their investments more. That is, Whited's study tests whether US publicly traded financially constrained firms have *capital stock adjustment hazards* that are lower than otherwise similar unconstrained firms and finds that this is indeed the case. DeAngelo, DeAngelo, and Stulz (2010) findings that firms conduct a

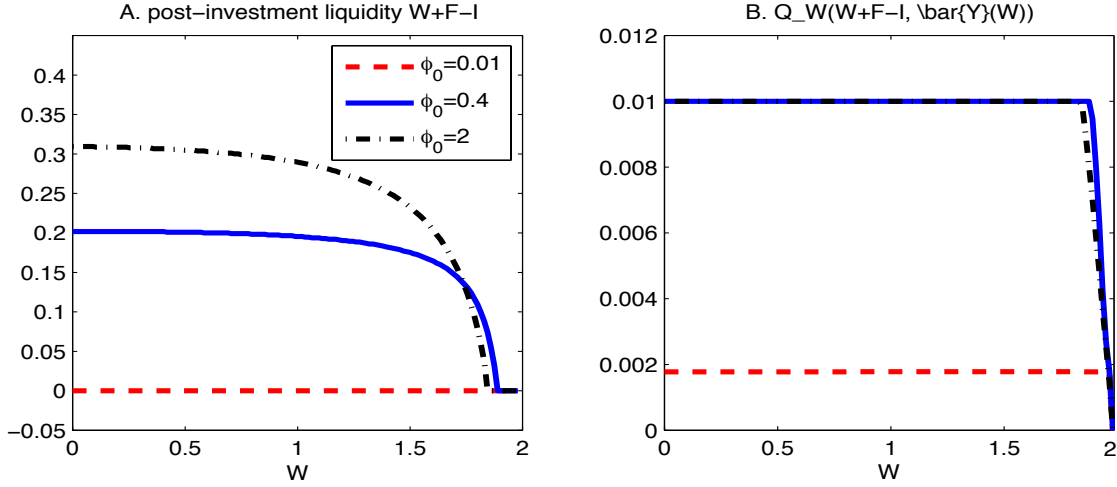


Figure 5: **Post-investment liquidity**  $W_{\tau_{i+}} = W_{\tau_i} + F_g(W_{\tau_i}) - I$ .

seasoned equity offering (SEO) mostly to fund a capital expenditure or a cash shortfall are also consistent with our results about the optimal timing of lumpy capital expenditures and financially constrained start-up firms' decision to tap external capital markets only to fund an investment outlay.

**The Financing Decision.** Figure 5 plots the post-investment liquidity  $W_{\tau_{i+}} = W_{\tau_i} + F_g(W_{\tau_i}) - I$  implied by the financing amount  $F_g(W)$  at the moment of exercising the growth option in the region  $W < I$ . First, consider the case with  $\phi_0 = 0.4$ . In the region  $W \geq 1.6$ , as  $W$  increases, the firm becomes increasingly less willing to issue equity to finance investment as the alternative of simply waiting for its internal funds to build up to cover the investment cost  $I$  is getting cheaper. As a result, the firm's investment hurdle increases significantly, approaching  $\infty$  as  $W \rightarrow I$ , as we discussed earlier. As a result, the amount of post-investment liquidity needed ever to cover operating losses is becoming increasingly small as internally generated cash flows (now and in the future) can easily finance the firm's potential liquidity need in the future. For example, post-investment liquidity  $W_{\tau_{i+}} = W_{\tau_i} + F_g(W_{\tau_i}) - I$  decreases from 0.202 at  $W_{\tau_i} = 0$  to 0 at  $W_{\tau_i} = 1.89$ . For  $W < 1.89$ , should the firm decided to issue equity, (e.g., at  $\bar{Y} = 1.592$  when  $W = 1$ ), it issues some more funds than needed to purely finance investment so that its post-investment liquidity is positive,  $W + F - I = 0.196$ . For lower values of  $\phi_0$ , e.g.,  $\phi_0 = 0.01$ , the post-investment liquidity is essentially zero for all levels of  $W$ , as the firm essentially has no need to hoard more liquidity than needed to cover the financing gap  $I - W$ .

Panel B shows that the marginal enterprise value of cash at the moment of investing,  $Q_W(W_{\tau_{i+}}, \bar{Y}(W_{\tau_i}))$ , is weakly lower than the proportional financing cost  $\phi_1 = 0.01$ . When  $Q_W(W_{\tau_{i+}}, \bar{Y}(W_{\tau_i})) = \phi_1 = 0.01$ , the firm's post-investment liquidity  $W_{\tau_{i+}}$  is positive, as it wants to hoard liquidity to cover the operating cost in the future. When  $Q_W(W_{\tau_{i+}}, \bar{Y}(W_{\tau_i})) < 0.01$ ,  $F_g(W_{\tau_i}) = I - W_{\tau_i}$  and the post-investment liquidity  $W_{\tau_{i+}} = 0$ .

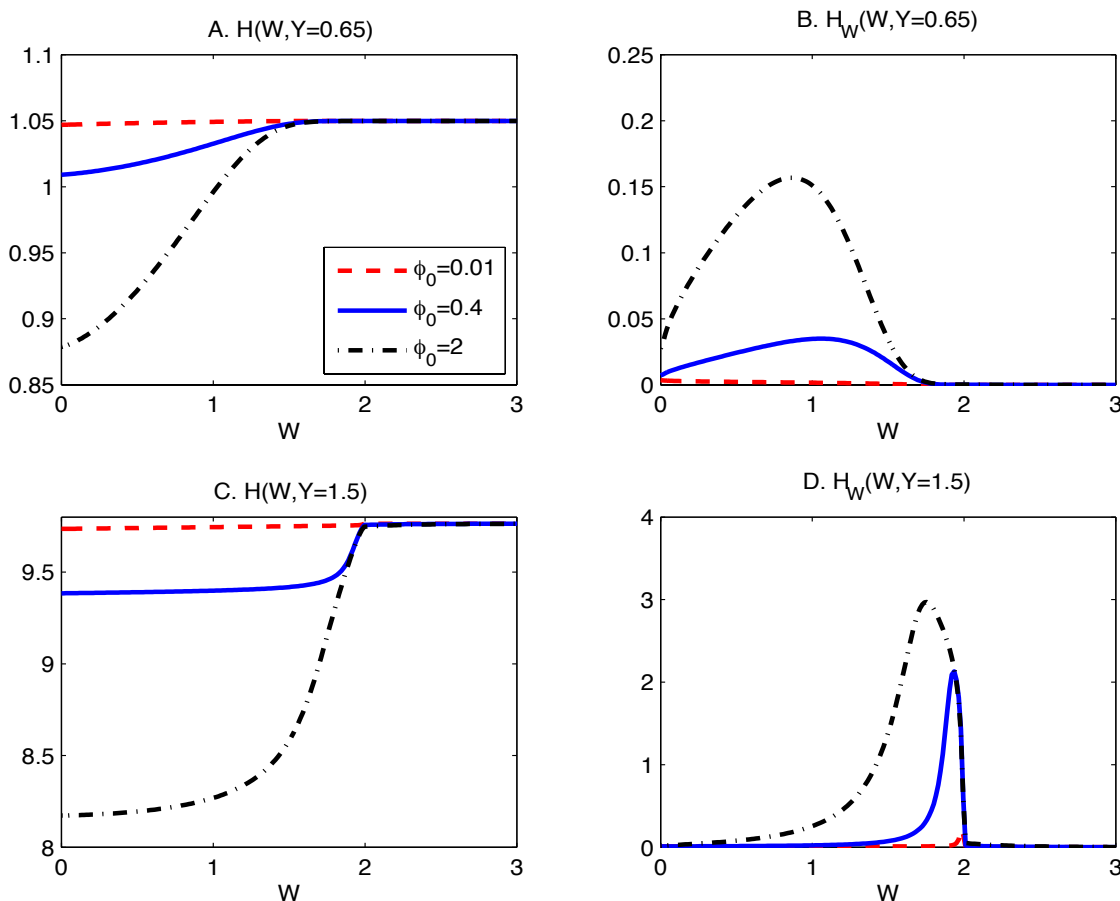


Figure 6: The firm's enterprise value in the startup phase,  $H(W, Y)$ , and the marginal value of liquidity  $H_W(W, Y)$  for three levels of fixed costs  $\phi_0 = 0.01, 0.4, 2$ .

**The Value of the Start-up Firm.** Figure 6 plots the firm's *enterprise value*  $H(W, Y)$  in the start-up phase and the corresponding marginal value  $H_W(W, Y)$  for  $Y = 0.65, 1.5$  and the same three fixed cost of issuing equity ( $\phi_0 = 0.01, 0.4, 2$ ).

With a small fixed financing cost,  $\phi_0 = 0.01$ ,  $H(W, Y)$  is increasing and concave in  $W$  as the firm is financially constrained and needs cash in order to invest. In contrast, with fixed financing costs, for both  $\phi_0 = 0.4$  and  $\phi_0 = 2$  cases, the (net) marginal value of cash  $H_W(W, Y)$  first increases with  $W$  and then decreases with  $W$  as  $W$  becomes sufficiently large. For example, consider the case with  $\phi_0 = 0.4$  and  $Y = 0.65$ ,  $H_W(W, 0.65)$  is upward sloping in  $W$  and reaches its highest value 0.035 around  $W = 1.05$ , and then declines as  $W$  decreases, effectively approaching 0 as  $W$  exceeds 1.85.

Why is the firm's growth option value  $H(W, Y)$  convex in  $W$  in the region  $W < 1.05$  for  $Y = 0.65$ ? Intuitively, if the firm was able to take a mean-preserving spread with  $W$  in that region, it would be better off, as either it accumulates internal funds faster or upon incurring losses, the firm gets closer to issue external equity to finance the exercising of its



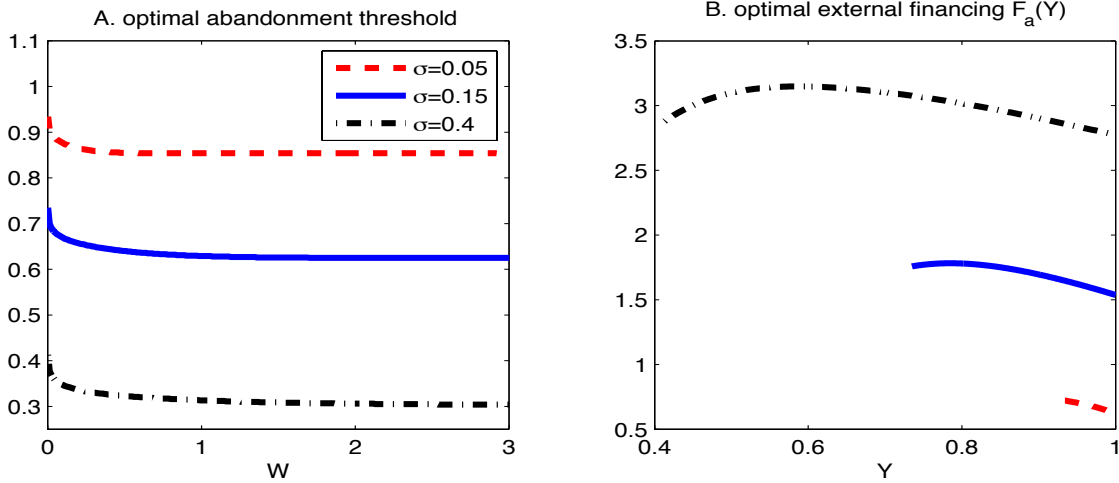


Figure 7: **The liquidation hurdle  $\underline{Y}(W)$  and financing decision  $F$  in the mature phase.** The financing cost parameters are  $\phi_0 = 0.4$  and  $\phi_1 = 0.01$ .

investment option. This additional benefit ameliorates the costly delay of the investment exercising decision.

In contrast, in the region where  $W > 1.05$ , the financially constrained firm has sufficiently high liquidity and hence has much weaker incentives to tap external financing and prefers waiting until the time when it has accumulated sufficient internal funds to finance investment. Therefore, the marginal value of cash  $H_W(W, Y)$  decreases with  $W$  in that region.

## 7 Comparative statics

How is firm value and the marginal value of cash affected by changes in the underlying volatility of income growth  $\sigma$ ? And, how are the firm's investment and abandonment decisions affected by changes in the firm's operating cost  $Z$ ? We turn to these questions in this section, by considering first the effects of changes in volatility.

### 7.1 Income Growth Volatility $\sigma$

One of the main results of (real) option theory is that the value of the option increases with underlying asset volatility  $\sigma$ . Does this result continue to hold for a financially constrained firm? As Figures 7-10 reveal, this result is robust to the introduction of external financing costs  $\Phi(F)$ . These figures plot respectively the mature firm value  $Q(W, Y)$ , the marginal value of cash  $Q_W(W, Y)$ , the start-up real option value  $H(W, Y)$  and the (start-up) marginal value of cash  $H_W(W, Y)$  as a function of  $W$  for respectively three levels of volatility ( $\sigma = 0.05, 0.15, 0.4$ ) and two realizations of productivity shock ( $Y = 0.65, 1.5$ ). Figure 9 also plots  $Q(W, Y)$  and  $Q_Y(W, Y)$  along the  $Y$  axis for two levels of cash holdings ( $W = 0, 2$ ). We

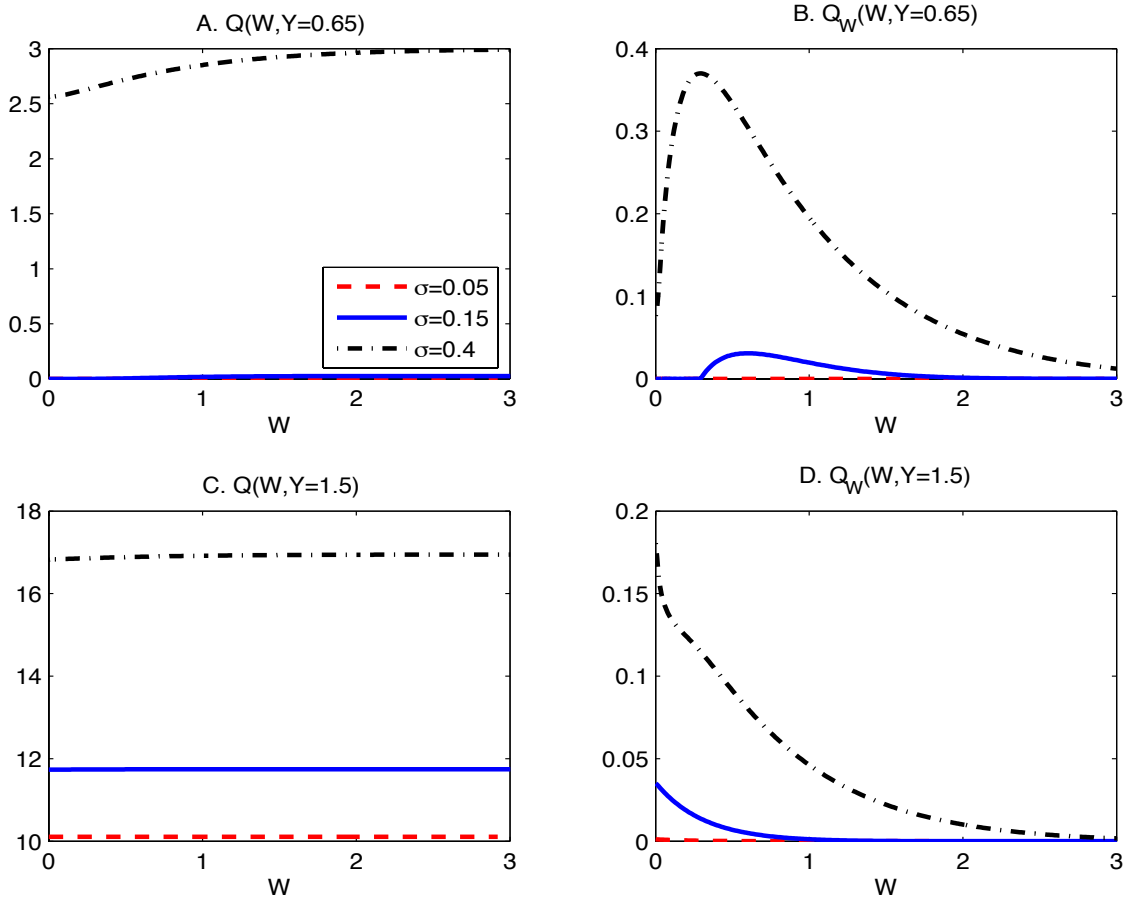


Figure 8: Enterprise value after the real option exercise,  $Q(W, Y)$ , and the marginal value of liquidity wealth  $Q_W$  for three levels of volatility  $\sigma = 0.05, 0.15, 0.4$ .

also plot the optimal abandonment policy  $\underline{Y}(W)$  and external financing policy in the mature phase in Figure 7 for the three levels of volatility  $\sigma = 0.05, 0.15, 0.4$ . Panel A of Figure 7 shows that the hurdle  $\underline{Y}(W)$  for abandoning the asset is higher the lower is volatility  $\sigma$ . For example, we have  $\underline{Y}(0; \sigma = 0.05) = 0.93 > \underline{Y}(0; \sigma = 0.15) = 0.74 > \underline{Y}(0; \sigma = 0.4) = 0.41$ . With lower asset volatility the firm's operating earnings prospects are less likely to improve and the firm faces a lower upside, so that it is less eager to continue operations. Panel B of Figure 7 shows that the firm raises more funds when it chooses to continue the higher is volatility: For example at  $Y = 1$  the firm raises an amount  $F_a(1; \sigma = 0.05) = 2.78 > F_a(1; \sigma = 0.15) = 1.54 > F_a(1; \sigma = 0.4) = 0.62$ . The reason is that with higher asset volatility the firm is more likely to end up in a situation where it accumulates such large losses that it needs to return to capital markets or abandon the asset. To reduce that risk the firm builds a higher precautionary cash buffer each time it returns to capital markets. Figure 8 plots the mature firm value  $Q(W, Y)$  and the marginal value of cash  $Q_W(W, Y)$  for two realizations of productivity shock ( $Y = 0.65, 1.5$ ). Panels A and C of Figure 8 simply

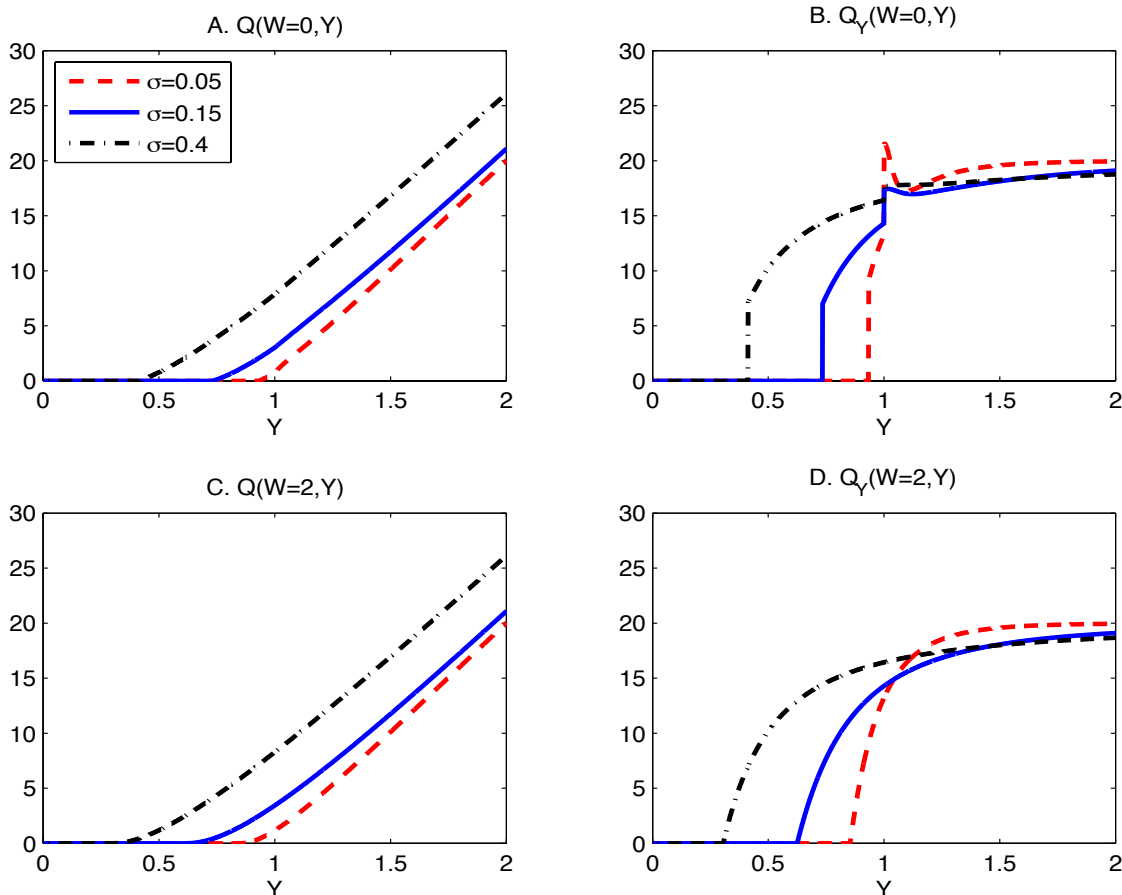


Figure 9: Enterprise value after the real option exercise,  $Q(W, Y)$ , and the marginal value of revenue  $Q_Y$  for three different levels of volatility  $\sigma = 0.05, 0.15, 0.4$ .

report that firm value is greater the higher is asset volatility. This is true even though the firm faces external financing costs when it runs out of cash. Panel B of Figure 8 shows that the marginal value of cash  $Q_W(W, 0.65)$  is non-monotonic in  $W$ . Obviously, when the abandonment option is out of the money there is no added value in keeping cash in the firm. This is why  $Q_W(W, 0.65) = 0$  for  $\sigma = 0.05$ , and why  $Q_W(W, 0.65) = 0$  for  $W \leq 0.29$  when  $\sigma = 0.15$ . Thus, when the firm approaches the point where it may abandon its asset cash has less value. But, when the firm is profitable ( $Y = 1.5$ ) then more cash  $W$  simply means that the firm is less likely to tap costly external capital markets. This is why  $Q_W(W, 1.5)$  is always decreasing in  $W$ , as Panel D of Figure 8 shows. Also, when the firm is profitable it can accumulate retained earnings, so that any dollar in retained cash is worth less. This is why for  $\sigma = 0.4$ ,  $Q_W(0.3, 1.5) = 0.11$ , which is much lower than  $Q_W(0.3, 0.65) = 0.37$ .

Figure 9 plots the firm's enterprise value  $Q(W, Y)$  and its sensitivity with respect to earning  $Y$ ,  $Q_Y(W, Y)$ , along the  $Y$  axis for two values of internal funds ( $W = 0, 2$ ). Panels A and C of Figure 9 show that firm value is increasing in  $Y$  and that  $Q(W, Y)$  is higher

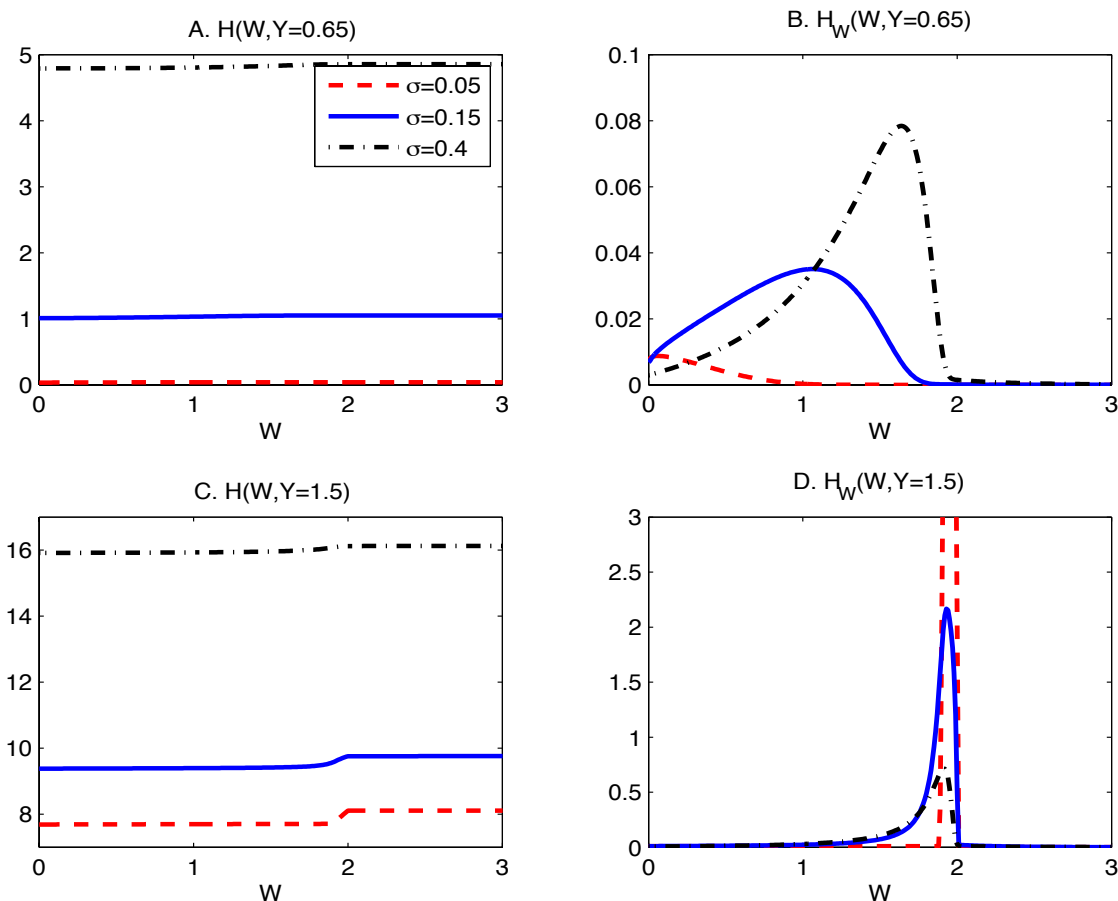
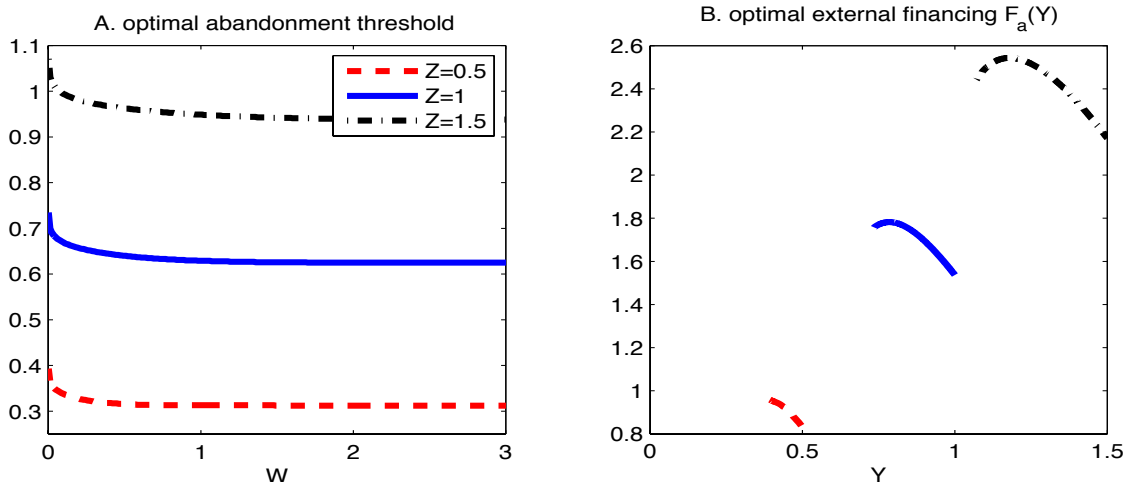


Figure 10: Real option value  $H(W, Y)$  and the marginal value of liquid wealth  $H_W$  for three different levels of volatility  $\sigma = 0.05, 0.15, 0.4$ .

the higher the asset volatility. This is altogether not surprising. The results in Panels B and D of Figure 9 are more interesting. They show how  $Q_Y(W, Y)$  is affected by internal funds  $W$  and asset volatility  $\sigma$ . The basic message from these results is that higher volatility raises marginal value of revenue when the firm is making losses ( $Y < 1$ ), given that higher volatility means higher option value. But when the firm is sufficiently profitable ( $Y \gg 1$ ) the effects of asset volatility on marginal value of revenue are reversed. Note finally, that a liquidity shortage ( $W = 0$ ) introduces a non-convexity around the break-even point  $Y = 1$ : as soon as the firm becomes profitable there is a drop in marginal value of revenue.

Figure 10 plots the start-up value  $H(W, Y)$  and marginal value of cash  $H_W(W, Y)$  for two realizations of productivity shock ( $Y = 0.65, 1.5$ ). Panels A and C of Figure 10 not surprisingly show that start-up value is also increasing in asset volatility. Again, the results in Panels B and D of Figure 10 are more interesting. They show that the marginal value of cash  $H_W(W, 0.65)$  is non-monotonic in  $W$ . But, the reason now is that when the firm approaches the point  $I = 2$ , where it could entirely finance its investment using internal funds

Figure 11: **The liquidation hurdle  $\underline{Y}(W)$  and financing decision in the mature phase.** The first-best liquidation (abandonment option exercising) hurdle is  $Y_a^* = 0.312, 0.625, 0.938$ . The liquidation boundary at  $W = 0$  is  $\underline{Y}(0) = 0.393, 0.735, 1.071$  for  $Z = 0.5, 1, 1.5$ . The liquidation boundary is monotonically decreasing with liquidity holding  $W$ .



each additional dollar is increasingly valuable. In sum, the non-monotonicity in  $H_W(W, Y)$  is simply a reflection of the non-monotonicity of the investment hurdle  $\bar{Y}(W)$  illustrated in Figure 4.

## 7.2 Operating cost $Z$ .

In a Modigliani-Miller world with no financial frictions operating costs  $Z$  can be netted out of gross operating revenues without any loss of generality. This is why most existing real option analyses do not explicitly model operating costs. When firms face external financing costs, however, it is important to model operating costs explicitly as changes in operating costs introduce novel effects through changes in *operating leverage*. Figure 11 plots the optimal abandonment policy  $\underline{Y}(W)$  and external financing policy in the mature phase in for the three operating cost levels  $Z = 0.5, 1, 1.5$ . Panel A of Figure 11 shows that the hurdle  $\underline{Y}(W)$  for abandoning the asset is, not surprisingly, higher the higher is  $Z$ . With higher costs the firm's net earnings prospects are reduced, so that it is less likely to continue operations. Panel B of Figure 11 shows that the firm raises more funds when it chooses to continue the higher are operating costs. The reason is that with higher costs the firm has higher operating leverage and is, therefore, more likely to end up with large losses that force it to return to capital markets or abandon the asset. The firm therefore builds a higher precautionary cash buffer to reduce this risk. Figure 12 plots the mature firm value  $Q(W, Y)$  and the marginal value of cash  $Q_W(W, Y)$  for two realizations of productivity shock ( $Y = 0.65, 1.5$ ). As one would expect, Panels A and C of Figure 12 show that firm value is greater the lower are operating

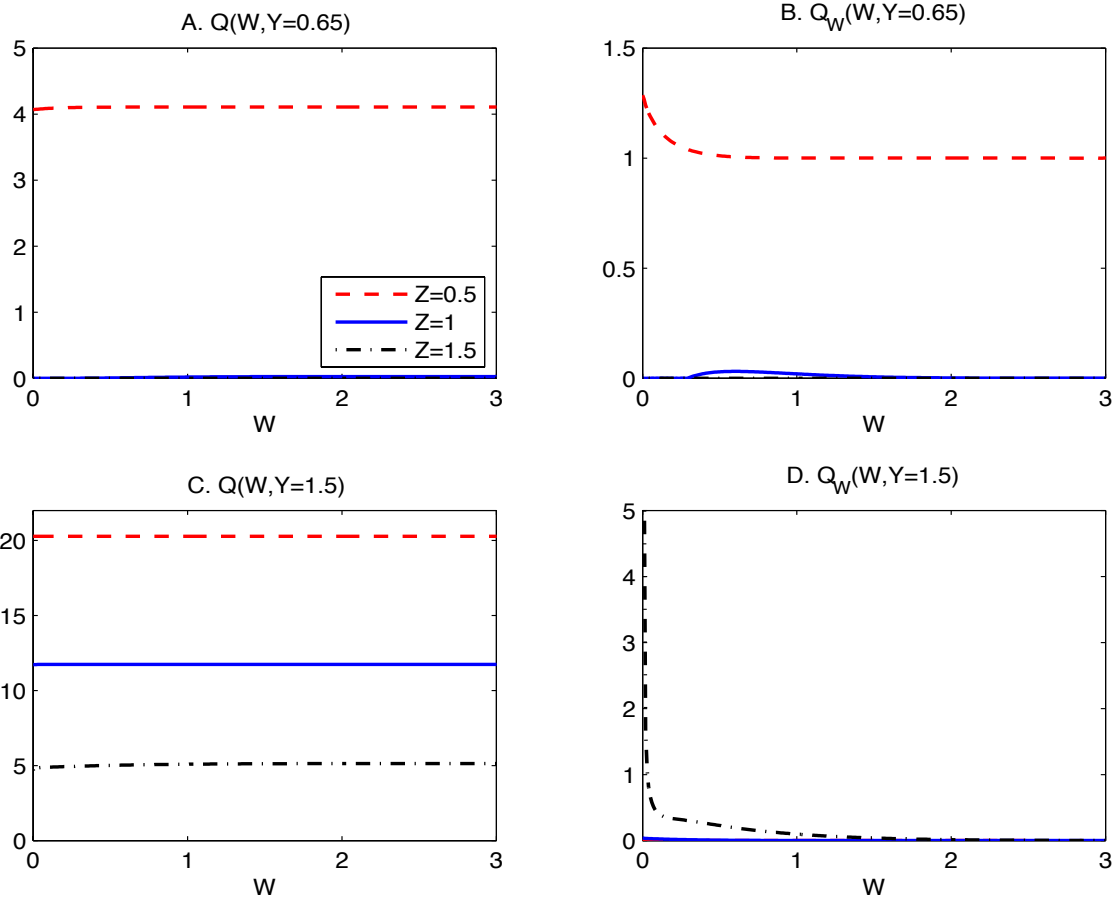


Figure 12: Enterprise value after the real option exercise,  $Q(W, Y)$ , and the marginal value of liquidity wealth  $Q_W$  for three levels of operating cost  $Z = 0.5, 1, 1.5$ .

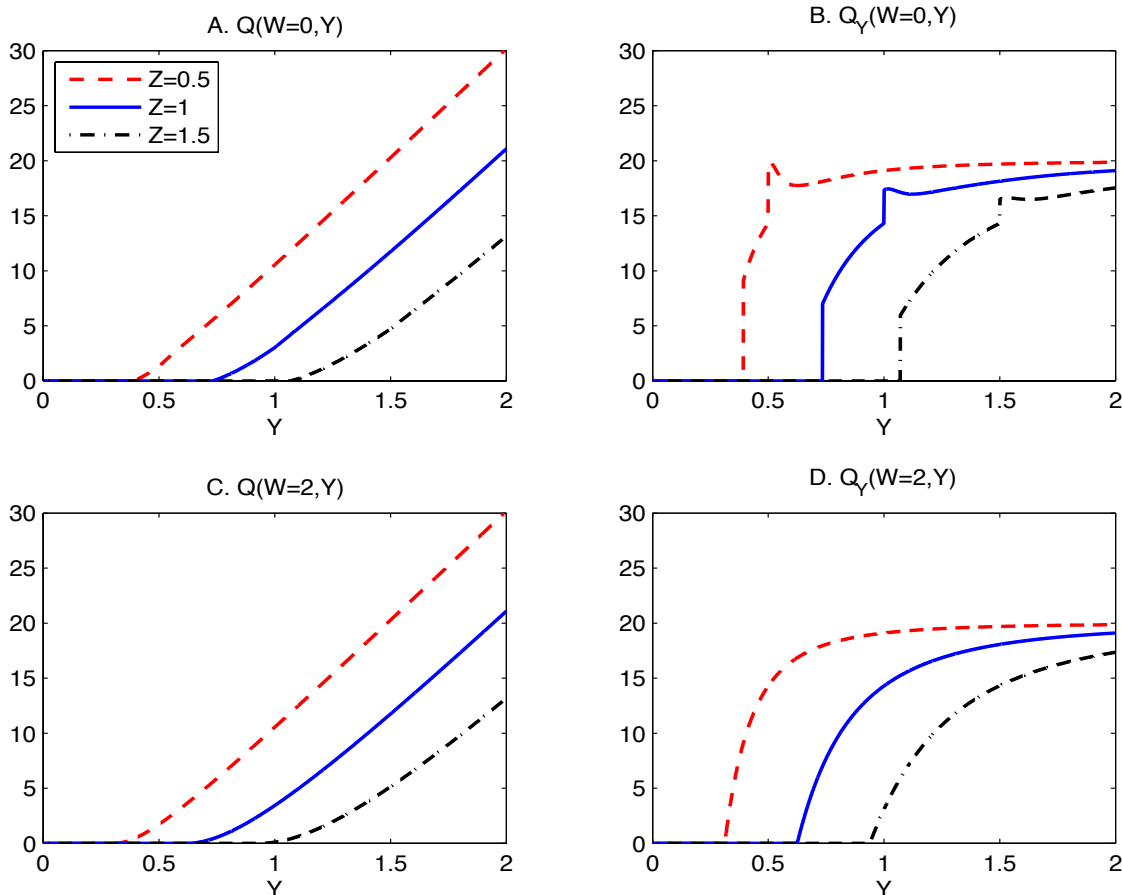


Figure 13: Enterprise value after the real option exercise,  $Q(W, Y)$ , and the marginal value of revenue  $Q_Y$  for three different levels of operating cost  $Z = 0.5, 1, 1.5$ .

costs. Panels B and D of Figure 12 show that the marginal value of cash  $Q_W(W, 0.65)$  can be non-monotonic in  $W$  for the same reason as before: when the abandonment option is out of the money there is no added value in keeping cash in the firm, but when it is in the money then the marginal value of cash is decreasing in  $W$ . Figure 13 plots the mature firm value  $Q(W, Y)$  and its sensitivity with respect to earning  $Y$ ,  $Q_Y(W, Y)$  along the  $Y$  axis for two values of internal funds ( $W = 0, 2$ ). Panels A and C of Figure 13 confirm the obvious: firm value is increasing in  $Y$  and  $Q(W, Y)$  is higher the lower the operating cost  $Z$ . The results in Panels B and D of Figure 13 show the subtle effects of internal funds  $W$  on the marginal value of revenue. When the firm has a sufficiently large cash buffer ( $W = 2$  the marginal value of revenue is increasing in  $Y$ ), but when it runs out of cash ( $W = 0$ ) the same non-convexity around the break-even point  $Y = Z$  as in Figure 9 appears, and the marginal value of revenue is decreasing in  $Y$  as soon as the firm becomes profitable. Figure 14 plots the start-up value  $H(W, Y)$  and marginal value of cash  $H_W(W, Y)$  for two realizations of the

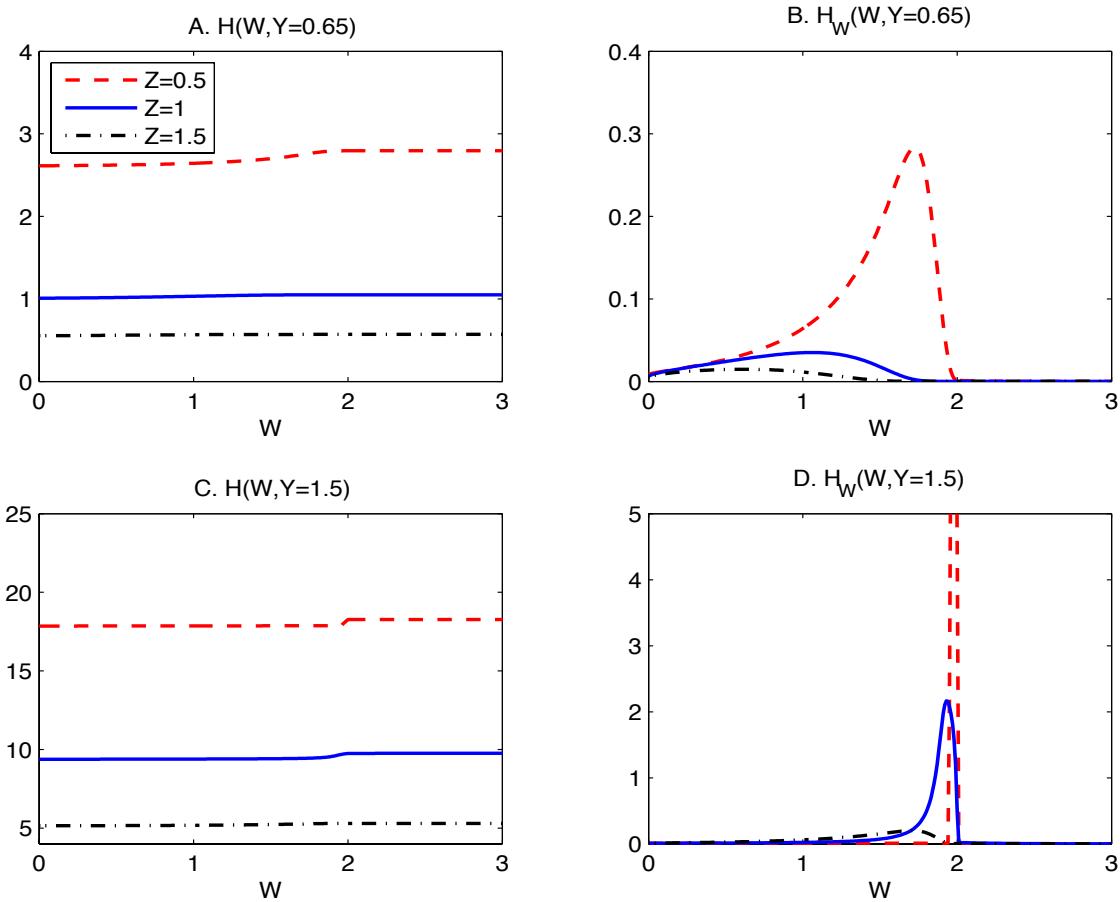


Figure 14: Real option value  $H(W, Y)$ , and the marginal value of liquidity wealth  $H_W$  for three levels of operating cost  $Z = 0.5, 1, 1.5$ .



productivity shock ( $Y = 0.65, 1.5$ ). Except for the lack of smoothness around  $W = I = 2$ , Panels A and C of Figure 14 again show the obvious: the start-up value of the firm is decreasing in operating costs  $Z$ . Panels B and D of Figure 14 show that the marginal value of cash  $H_W(W, 0.65)$  is non-monotonic in  $W$  for the same reason as in Figure 10: when the firm approaches the point  $I = 2$  each additional dollar is increasingly valuable. But, note that this non-monotonicity is less pronounced as operating costs  $Z$  rise. The reason is that with a higher operating cost  $Z$  the investment option is further out of the money, hence the marginal value of liquidity is lower.

## 8 Sequential Investment

We now generalize our model to allow for two rounds of investment options. Accordingly, we refer to stage 0 as the *start-up phase* when the firm has not acquired any productive asset, stage 1 as the *growth phase* that starts when the firm has exercised its first growth option and hence operates the its asset in place, but still holds the second growth option, and stage 2 as the *mature phase* that starts after the second investment option has also been exercised. As before, due to operating leverage, the firm may exercise its abandonment option in either the growth phase or the mature phase. The firm jointly determines its two sequential growth option exercising, its abandonment option exercise, and the corresponding liquidity management problem.

This extension allows us to analyze two novel effects. The first is the change in incentives to exercise its first investment option caused by the prospect of acquiring valuable operating cash-flows that help relax the internal funding constraint to finance the exercising cost of the second growth option and the liquidity needs to finance subsequent operations of assets in place. The second is how incentives to exercise the later investment option and other corporate decisions are modified by operating cash-flow risk from the first productive asset.

A striking new result is that when the firm faces multiple rounds of growth options it may choose to invest in the earlier opportunities sooner than under the first-best benchmark. In other words, the firm optimally engages in a form of over-investment relative to the first-best as a way of building up its liquidity hoarding capacity which helps mitigate the incentives to delay the second growth option. That is, accelerated investment timing in the first phase may be in the firm's interest if the benefit of over-investing (via accelerating the investment timing) outweighs the costs of under-investment (due to delayed investment timing decision for the follow-up growth option).

We denote by  $Z_n$  the firm's operating cost after exercising the  $n$ th growth option, where  $n \in \{1, 2\}$ . Correspondingly, we denote the stochastic revenue process generated by the productive asset by  $x_n Y_t$ , where  $Y_t$  is given in (1) and where  $x_1 < x_2$  denotes the production capacity after exercising respectively the first and second investment option. That is, upon the exercising of the second growth option, the firm's operating revenue increases by  $\Delta_x Y$  where  $\Delta_x = x_2 - x_1$  and the operating cost increases by  $\Delta_Z = Z_2 - Z_1$ . Intuitively, we may interpret  $\Delta_x Y - \Delta_Z$  as the additional revenue generated by the second asset in place (obtained by the exercising of the second growth option.)

We also denote by  $\tau_L^n$  the abandonment time of the  $n$ th productive asset, where  $n \in \{1, 2\}$

and  $0 \leq \tau_L^1 \leq \tau_L^2$ . In the mature phase of sequential investment case, we obtain the same firm value, same dynamics for liquidity, and the same abandonment decision as the baseline model by replacing the cash flow with  $x_2Y$  and the operation cost with  $Z_2$  in the baseline model. The main changes to the analysis are thus in stages 0 and 1, to which we now turn. During stage 0 the liquidity dynamics are the same as in the baseline model for the start-up phase and are given in (3). As for stage 1, the firm's liquidity  $W$  accumulates as follows:

$$dW_t = (rW_t + x_1Y_t - Z_1)dt + dC_t, \quad W_t \geq 0. \quad (42)$$

Before solving for the financially constrained firm's value and optimal investment decisions it is helpful to characterize the first-best benchmark where the firm faces no external funding costs.

## 8.1 First-best Benchmark under Modigliani-Miller Assumptions

Let  $H^*(Y)$  denote the solution for the first-best enterprise value in stage 0,  $Q_n^*(Y)$  the solutions for the enterprise values in respectively stages  $n = 1, 2$ , and  $Y_{n,a}^*$  the optimal abandonment boundaries in stages  $n = 1, 2$ . Moreover, let  $Y_{n,i}^*$  denote the optimal investment hurdles of the  $n$ th growth options.

We can then write  $Q_1^*(Y)$  as the solution to the following ODE:

$$rQ_1(Y) = (x_1Y - Z_1) + \mu Y Q_1'(Y) + \frac{\sigma^2 Y^2}{2} Q_1''(Y), \quad (43)$$

subject to the value-matching and smooth-pasting boundary conditions for the investment and abandonment options:

$$\begin{aligned} Q_1(Y_{2,i}^*) &= Q_2^*(Y_{2,i}^*) - I_2, \\ Q_1'(Y_{2,i}^*) &= Q_2^{*'}(Y_{2,i}^*), \\ Q_1(Y_{1,a}^*) &= 0, \\ Q_1'(Y_{1,a}^*) &= 0. \end{aligned} \quad (44)$$

Note that equation (43) for  $Q_1(Y)$  is identical to equation (7) for  $Q(Y)$  except for the first term, which reflects the fact that the firm is not operating at full scale and only has one productive asset with capacity  $x_1$  and operating costs  $Z_1$ . The upper boundary  $Y_{2,i}^*$  at which the firm decides to scale up to full capacity is such that the growth-phase enterprise value  $Q_1(Y_{2,i}^*)$  is just equal to the mature-phase enterprise value  $Q_2^*(Y_{2,i}^*)$  net of investment costs  $I_2$ .

As we have established in Section 3.1, the mature-phase value  $Q_2^*(Y; Z_2) = Q^*(x_2Y; Z = Z_2)$  is given by:

$$Q_2^*(Y) = \left( \frac{x_2Y}{r - \mu} - \frac{Z_2}{r} \right) + \left( \frac{Y}{Y_{2,a}^*} \right)^\gamma \left( \frac{Z_2}{r} - \frac{x_2Y_{2,a}^*}{r - \mu} \right), \quad \text{for } Y \geq Y_{2,a}^*, \quad (45)$$

where

$$Y_{2,a}^* = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{r} \frac{Z_2}{x_2}. \quad (46)$$

After solving for  $Q_2^*(Y)$  we can move backwards to the growth phase and solve for  $Q_1^*(Y)$ . As can be readily verified, the solution for  $Q_1^*$  when  $Y_{2,i}^* \geq Y \geq Y_{1,a}^*$  takes the following form:

$$Q_1^*(Y) = \left( \frac{x_1 Y}{r - \mu} - \frac{Z_1}{r} \right) + \Psi_i(Y) \left( Q_2^*(Y_{2,i}^*) - I_2 + \frac{Z_1}{r} - \frac{x_1 Y_{2,i}^*}{r - \mu} \right) + \Psi_a(Y) \left( \frac{Z_1}{r} - \frac{x_1 Y_{1,a}^*}{r - \mu} \right), \quad (47)$$

with:

$$\Psi_i(Y) = \left( \frac{Y_{1,a}^{*\beta} Y^\gamma - Y_{1,a}^{*\gamma} Y^\beta}{Y_{1,a}^{*\beta} Y_{2,i}^{*\gamma} - Y_{1,a}^{*\gamma} Y_{2,i}^{*\beta}} \right) \quad \text{and} \quad \Psi_a(Y) = \left( \frac{Y_{2,i}^{*\gamma} Y^\beta - Y_{2,i}^{*\beta} Y^\gamma}{Y_{1,a}^{*\beta} Y_{2,i}^{*\gamma} - Y_{1,a}^{*\gamma} Y_{2,i}^{*\beta}} \right), \quad (48)$$

where  $\beta$  and  $\gamma$  are determined by the boundary conditions (44).

Finally, we can move backwards to the start-up phase and solve for  $H^*(Y)$ , which satisfies the ODE (15) and the boundary conditions:

$$\begin{aligned} H(Y_{1,i}^*) &= Q^*(Y_{1,i}^*) - I_1, \\ H'(Y_{1,i}^*) &= Q^{*'}(Y_{1,i}^*). \end{aligned}$$

Again, it is straightforward to verify that  $H^*(Y)$  takes the following form:

$$H^*(Y) = \left( \frac{Y}{Y_{1,i}^*} \right)^\beta (Q_1^*(Y_{1,i}^*) - I_1) \quad \text{for} \quad Y \leq Y_{1,i}^*. \quad (49)$$

## 8.2 General Solution under External Financing Costs

### 8.2.1 The Mature Phase

In Section 4, we established that a firm can be in one of three liquidity regions in the mature phase: *i*) a financially unconstrained region (when  $W$  is sufficiently large); *ii*) an interior financially constrained region, and; *iii*) an equity issuance/liquidation region. This obviously continues to be true for the mature phase in our more general setting. Following earlier analysis, we have that abandonment in the mature phase is an all or nothing decision, the solution for the firm's value  $P^{(2)}(W, Y; Z_2)$  in the abandonment phase is identical to that in section 4 for  $P(W, x_2 Y; Z = Z_2)$ .

### 8.2.2 The Growth Phase

The firm may also be in one of these three regions in the growth phase (stage 1). Proceeding as in section 4 we denote by  $\Lambda_1$  and  $\Lambda_2$  the lowest levels of liquidity needed to be permanently financially unconstrained for a firm that is respectively in the growth phase and in the mature phase. In the mature phase we have as before:

$$\Lambda_2 \equiv \frac{Z_2 - Y_{2,a}^*}{r} \equiv \frac{r - \gamma\mu}{r^2(1 - \gamma)} Z_2.$$

Similarly, the expression for  $\Lambda_1$  in the growth phase is:

$$\Lambda_1 \equiv \frac{Z_1 - x_1 Y_{1,a}^*}{r}. \quad (50)$$

**The Unconstrained Region:** When  $W \geq \max\{\Lambda_1, I_2 + \Lambda_2\}$  or  $Y \rightarrow \infty$  the firm will be financially unconstrained going forward even in the growth phase, and the value of the firm will be equal to the first-best value:  $P^{(1)}(W, Y) = Q_1^*(Y) + W$ .

**The Financially Constrained Region:** When  $W < \max\{\Lambda_1, I_2 + \Lambda_2\}$ , the firm's value  $P^{(1)}(W, Y)$  evolves according to the following HJB equation:

$$rP^{(1)}(W, Y) = (rW + x_1Y_1 - Z_1)P_W^{(1)}(W, Y) + \mu Y P_Y^{(1)}(W, Y) + \frac{\sigma^2 Y^2 P_{YY}^{(1)}(W, Y)}{2}. \quad (51)$$

It satisfies the following abandonment boundary conditions:

$$P^{(1)}(W, \underline{Y}_1(W)) = W,$$

and

$$P_Y^{(1)}(W, \underline{Y}_1(W)) = 0.$$

It also satisfies the financing boundary condition:

$$P^{(1)}(0, Y) = \begin{cases} P^{(1)}(F, Y) - F - \Phi(F), & \underline{Y}_1(0) < Y < Z_1, \\ 0, & Y \leq \underline{Y}_1(0). \end{cases} \quad (52)$$

where the optimal amount of funds raised  $F$  (net of issuing costs) is given by:

$$P_W^{(1)}(F, Y) = 1 + \Phi'(F) = 1 + \phi_1, \quad \underline{Y}_1(0) < Y < Z_1. \quad (53)$$

Finally,  $P^{(1)}(W, Y)$  also satisfies a set of investment boundary conditions, which are described as follows:

1. When  $W < I_2$  the investment boundary conditions for  $P^{(1)}(W, Y)$  are:

$$P^{(1)}(W, \bar{Y}_2(W)) = P^{(2)}(W + F_g^{(2)}(W) - I_2, \bar{Y}_2(W)) - F_g^{(2)}(W) - \Phi(F_g^{(2)}(W))$$

and

$$P_Y^{(1)}(W, \bar{Y}_2(W)) = P_Y^{(2)}(W + F_g^{(2)}(W) - I_2, \bar{Y}_2(W)),$$

with the corresponding optimal financing conditions:

$$P_W^{(2)}(W + F_g^{(2)}(W) - I_2, \bar{Y}_2(W)) \leq 1 + \Phi'(F_g^{(2)}(W)) \text{ and } F_g^{(2)}(W) \geq I_2 - W, \quad (54)$$

where  $F_g^{(2)}(W)$  is the optimal amount of external funds the firm raises in excess of  $I_2 - W$ .

2. When  $I_2 < W < \max\{\Lambda_1, I_2 + \Lambda_2\}$ , the investment boundary conditions for  $P^{(1)}(W, Y)$  become:

$$P^{(1)}(W, \bar{Y}_2(W)) = P^{(2)}(W - I_2, \bar{Y}_2(W)) \quad (55)$$

and

$$P_Y^{(1)}(W, \bar{Y}_2(W)) = P_Y^{(2)}(W - I_2, \bar{Y}_2(W)).$$

### 8.2.3 The Start-up Phase

Finally, the solution for the start-up phase is also divided into three regions depending on the firm's liquidity  $W$ . In the financially unconstrained region, where  $W \geq I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}$  the solution is:

$$G(W, Y) = H^*(Y) + W, \quad \text{for } W \geq I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}. \quad (56)$$

In the financial constraint region where  $W < I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}$  the firm's value  $G(W, Y)$  solves the HJB equation:

$$rG(W, Y) = rWG_W(W, Y) + \mu Y G_Y(W, Y) + \frac{\sigma^2 Y^2}{2} G_{YY}(W, Y), \quad (57)$$

and the following boundary conditions for investment must hold:

1. When  $W \geq I_1$ :

$$\begin{aligned} G(W, \bar{Y}_1(W)) &= P^{(1)}(W - I_1, \bar{Y}_1(W)), \\ G_Y(W, \bar{Y}_1(W)) &= P_Y^{(1)}(W - I_1, \bar{Y}_1(W)). \end{aligned}$$

2. When  $0 \leq W < I_1$ :

$$\begin{aligned} G(W, \bar{Y}_1(W)) &= P^{(1)}(W + F_g^{(1)}(W) - I_1, \bar{Y}_1(W)) - F_g^{(1)}(W) - \Phi(F_g^{(1)}(W)), \\ G_Y(W, \bar{Y}_1(W)) &= P_Y^{(1)}(W + F_g^{(1)}(W) - I_1, \bar{Y}_1(W)), \end{aligned}$$

where the optimal amount raised over and above  $I_1 - W$  from external markets  $F_g^{(1)}(W)$  is given by:

$$P_W^{(1)}(W + F_g^{(1)}(W) - I_1, \bar{Y}_1(W)) \leq 1 + \Phi'(F_g^{(1)}(W)) \quad \text{and} \quad F_g^{(1)}(W) \geq I_1 - W.$$

Finally,  $G(W, Y)$  must also satisfy the absorbing barrier condition  $G(W, 0) = W$ .

## 8.3 Results

As before we set the risk-free interest rate at  $r = 5\%$ , the expected earnings growth rate at  $\mu = 0$ , and the earnings growth volatility at  $\sigma = 15\%$ . We set  $x_1 = 1, Z_1 = 1$  and  $I_1 = 2$  in stage 1, (the same parameters as in the baseline model.) In addition, we assume that  $x_2 = 2, Z_2 = 2$  and  $I_2 = 5$ . These parameter values effectively make the second asset in place less valuable than the first asset in place, a form of decreasing returns to scale. For all the other parameters in stage 1 and 2, we keep them the same as those in the baseline model.

**To be completed.**

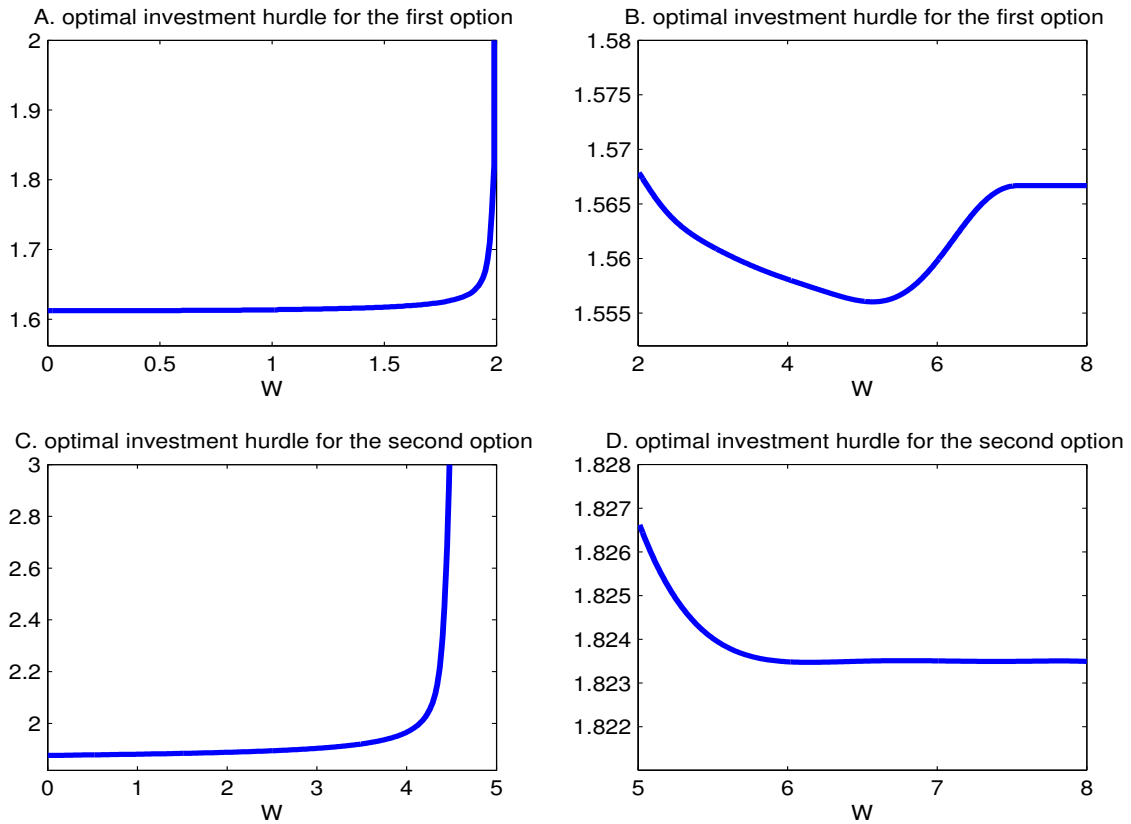


Figure 15: The optimal investment hurdle  $\bar{Y}_1(W)$  and  $\bar{Y}_2(W)$  as a function of pre-investment liquidity  $W$ .

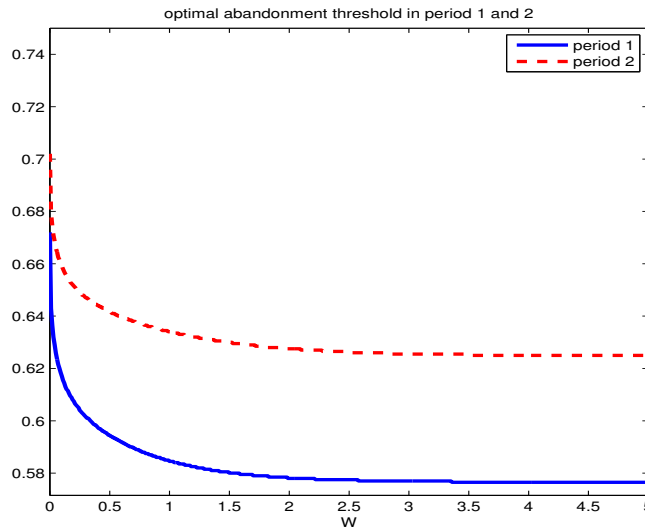


Figure 16: The optimal liquidation hurdle  $\underline{Y}_1(W)$  and  $\underline{Y}_2(W)$  for a financially constrained firm in stage 1 and 2.

## 9 Conclusion

We have shown that when firms face external financing costs, their dynamic corporate investment policy depends in a subtle way on the availability of internal funds. Our analysis also brings out the subtle interactions between the optimal timing of investment and the firm's sources of funds, whether external, internal, or from future retained earnings from investment. When the firm is able to accumulate more internal funds it may well be induced to slow down its investment even further as the prospect of being able to entirely avoid incurring external funding costs is closer. Our framework is also sufficiently parsimonious to be able to provide quantitatively plausible estimates of the value of the firm's real options and marginal value of cash, as well as its optimal investment and external funding policy. We have, however, left several important features out of our analysis. In future work we plan to incorporate a richer life-cycle framework for the firm, with in particular multiple rounds of growth options. We also plan to extend the framework by including access by the firm to a credit line as an alternative source of liquidity. Finally, we have left out the capital structure margin from our analysis. Sundaresan, Wang, and Yang (2013) explore this dimension of corporate financial policy in a dynamic real options framework but abstracting from the accumulation of internal funds.

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