A Unified Theory of Tobin's q, Corporate Investment, Financing, and Risk Management*

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Abstract

This paper proposes an analytically tractable dynamic model of corporate investment and risk management for a financially constrained firm. Following Froot, Scharfstein, and Stein (1993), we define a corporation's risk management as the coordination of investment and financing decisions. In our model, corporate risk management is a combination of internal liquidity management, financial hedging, investment, and payout decisions. We determine the firm's optimal risk management policies as functions of the following key parameters: 1) the firm's earnings growth and cash flow risk; 2) the external cost of financing; 3) the firm's liquidation value; 4) the opportunity cost of holding cash; 5) investment adjustment and asset sales costs; and 6) the return and covariance characteristics of hedging assets the firm can invest in. The optimal cash inventory policy involves two endogenous barriers and the continuous adjustment in investment and hedging positions in between the barriers. Cash is paid out to shareholders only when the cash-capital ratio hits the upper barrier, and external funds are raised only when the firm has depleted its cash. Several new insights emerge from our analysis. For example, we find that the relation between marginal q and investment differs depending on whether cash or credit is the marginal source of financing. We also demonstrate the distinct and complementary roles that cash management and derivatives play in risk management.

I. Introduction

When firms face external financing costs, they must deal with complex and closely intertwined investment, financing, and risk management decisions. Although the interconnection among these policies is well appreciated in theory, how to translate this observation into day-to-day risk management and investment policies still remains largely to be determined. Simple questions such as when/how corporations should reduce their cash holdings, or when/how they should replenish their dwindling cash inventory are still not precisely understood. Similarly, the questions of which risks the corporation should hedge and by how much, or to what extent holding cash inventory is a substitute for financial hedging through futures, swaps and derivatives are not well understood.

Our goal is to propose the first elements of a tractable dynamic economic framework, in which corporate investment, asset sales, cash inventory, equity financing, credit line, and dynamic hedging policies are characterized analytically for a "financially constrained" firm (that is, a firm facing external financing costs). The baseline model we propose introduces only the essential building blocks, which are: i) the workhorse neoclassical q model of investment[1] featuring a constant investment opportunity set (Hayashi (1982)); ii) time-invariant external financing costs and cash carry costs, so that the firm's financing opportunity set is also constant; iii) four financial instruments: cash, equity, credit line, and derivatives (e.g. futures). This parsimonious model already captures many situations that firms face in practice (at least as a first approximation) and yields a rich set of prescriptions.

A first important result that emerges from our analysis is that with external financing costs the firm's investment is no longer determined by equating the marginal cost of investing with marginal q, as in the neoclassical Modigliani-Miller (MM) model (with no fixed adjustment costs for investment).[2] Instead, corporate investment is determined by the following first-order condition:

$$\text{marginal cost of investing} = \frac{\text{marginal } q}{\text{marginal cost of financing}}.$$

[1] Tobin (1969) first introduces q as the ratio of firm market value to the replacement cost of its capital stock.

[2] See Abel and Eberly (1994) for a general specification of the q theory of investment under the neoclassic setting with both fixed and variable costs. Fixed costs of investment give rise to 'inaction' regions and generate real options for the firm as in McDonald and Siegel (1986) and Dixit and Pindyck (1994).

In other words, investment of a financially constrained firm is determined by the *ratio of marginal q to the marginal cost of financing*.[3] When firms are flush with cash, the marginal cost of financing is approximately one, so that this equation is approximately the same as the one under MM-neutrality. But when firms are close to financial distress, the marginal cost of financing may be much larger than one so that optimal investment may be far lower than the level predicted under MM-neutrality.

The above first order condition also implies that the relation between marginal q and investment differs depending on whether cash or credit is the marginal source of financing. We show that when the marginal source of financing is cash, both marginal q and investment increase with the firm's cash holdings, as more cash makes the firm less financially constrained. In contrast, when the marginal source of financing is a credit line, we show that marginal q and investment move in opposite directions: marginal q increases with the firm's leverage, while investment decreases with leverage. Indeed, an increase in investment helps relax the firm's borrowing constraint by adding capital that may be pledged as collateral against the credit line. This explains why marginal q increases with leverage. However, the more debt the firm has, the more aggressively it cuts investment to delay incurring equity issuance costs. We thus simultaneously observe an increasing marginal q schedule and a decreasing investment schedule as the firm takes on more debt.

A second important result concerns the firm's optimal cash-inventory policy. Much of the empirical literature on firms' cash holdings tries to identify a *target cash-inventory* for a firm by weighing the costs and benefits of holding cash.[4] The implicit idea is that this target level helps determine when a firm should increase its cash savings and when it should dissave.[5] Our analysis, however, shows that the firm's cash-inventory policy is much richer, as it involves a combination of a *double-barrier policy*, as in Miller and Orr (1966), and the continuous management of cash reserves in between the barriers through adjustments in investment, asset sales, as well as the firm's hedging positions. When cash holdings are higher, the firm invests more and saves less,

[3]This first-order condition has also been derived in Hennessy, Levy, and Whited (2007).

[4]See Almeida, Campello, and Weisbach (2004, 2008), Faulkender and Wang (2006), Khurana, Martin, and Pereira (2006), and Dittmar and Mahrt-Smith (2007).

[5]Recent empirical studies have found that corporations tend to hold more cash when their underlying earnings risk is higher or when they have higher growth opportunities (see e.g. Opler, Pinkowitz, Stulz, and Williamson (1999) and Bates, Kahle, and Stulz (2008)).

as the marginal value of cash is smaller. When the firm is approaching the point where its cash reserves are depleted, it optimally scales back investment and may even engage in asset sales. This way the firm can postpone or avoid raising costly external financing.

At the endogenous upper barrier of the cash-capital ratio it is optimal for the firm to pay out cash, and at the lower barrier the firm either raises more external funds or closes down. This lower barrier is attained when the firm runs out of cash and credit, as carrying cash and accessing credit are costly. Moreover, using internal funds (cash) to finance investment both lowers the cash carry costs and defers external financing costs. Thus, with a constant investment/financing opportunity set our model generates a dynamic pecking order of financing between internal and external funds. The stationary cash-inventory distribution from our model shows that firms respond to the financing constraints by optimally managing their cash holdings so as to stay away most of the time from financial distress situations.

A third general result is that in the presence of external financing costs, firm value is sensitive to both idiosyncratic and systematic risk. To limit its exposure to systematic risk, the firm can engage in dynamic hedging via derivatives (such as oil or currency futures). To mitigate the impact of idiosyncratic risk, it can manage its cash reserves by modulating its investment outlays and asset sales, and also by delaying or moving forward its cash payouts to shareholders.

Our model thus integrates two channels of risk management, one via a state non-contingent vehicle (cash), the other via state-contingent instruments (derivatives). The main benefit of reducing the firm's exposure to systematic risk through financial hedging is the reduction in the firm's need to hold costly cash inventory. Derivatives and cash thus play complementary roles in risk management. However, when dynamic hedging involves higher transactions costs, such as tighter margin requirements, we also show that the firm reduces its hedging positions and uses cash as a substitute.

A fourth result concerns the relation between the firm's beta and its cash holdings. To the extent that the beta of a financially constrained firm reflects the firm's exposure to both idiosyncratic and systematic risk, it should be higher than the beta of an unconstrained (first-best) firm, which reflects only the firm's exposure to systematic risk. This intuition is valid in a static setting. However, we show that in a dynamic setting in which firms actively manage their cash holdings, a financially constrained firm can actually have a lower beta than an unconstrained firm. The intuition is as follows. In anticipation of future financing costs, a financially constrained firm is likely to hold a significant proportion of its assets in cash, which has a zero beta, while an unconstrained firm does not hold any cash.

Despite the potential technical complexity of an analysis of dynamic corporate risk management, our model is sufficiently simple that we are able to provide a precise analytical characterization of the firm's optimal policy. We can thus give concrete prescriptions for how a firm should manage its cash reserves and choose its investment, financing, hedging, and payout policies, given its underlying production technology, investment opportunities, financing costs, and market interest rates. Moreover, we are able to provide a number of interesting comparative statics results. We can also simulate the stationary distributions for any economic variable of interest such as the firm's cash-capital ratio, investment-capital ratio, firm value-capital ratio, and the marginal value of financing.

There is only a handful of theoretical analyses of firms' optimal cash, investment and risk management policies. A key first contribution is by Froot, Scharfstein, and Stein (1993), who develop a static model of a firm facing external financing costs and risky investment opportunities.[6] Our dynamic risk management problem uses the same contingent-claim methodology as in the dynamic capital structure/credit-risk models of Fischer, Heinkel, and Zechner (1989) and Leland (1994), but unlike these theories we explicitly model the wedge between internal and external financing of the firm and the firm's cash accumulation process. Our model also extends these latter theories by introducing capital accumulation and thus integrates the contingent-claim approach with the dynamic investment/financing literature. The following contributions in that latter literature are most closely related to ours.

Gomes (2001) is an important early contribution on dynamic corporate investment with external financing costs. His model, however, does not allow for cash inventory management. Hennessy

[6]See also Kim, Mauer, and Sherman (1998). Another more recent contribution by Almeida, Campello, and Weisbach (2008) extends the Hart and Moore (1994) theory of optimal cash holdings by introducing cash flow and investment uncertainty in a three-period model.

and Whited (2005, 2007) numerically solve and estimate discrete-time dynamic capital structure models with investment for financially constrained firms. They explicitly model taxes and allow for stochastic investment opportunities, but have no adjustment costs for investment.[7] Hennessy, Levy, and Whited (2007) characterize a similar investment first-order condition as ours for a financially constrained firm. They do not model fixed costs of equity issuance, and their analysis focuses on firms at the payout or equity issue margins. Using a model related to Hennessy and Whited (2005, 2007), Riddick and Whited (2008) show that saving and cash flow can be negatively related after controlling for q, because firms use cash reserves to invest when receiving a positive productivity shock.[8]

Our paper provides the first analysis of dynamic risk management, combining cash management and dynamic hedging, and the coordination between the firm's investment, financing and payout decisions. Unlike the existing dynamic investment/financing literature, our model exploits the analytical simplicity of a homogeneous model (linear in the capital stock), for which a complete analytical characterization of the firm's optimal investment and financing policies, as well as its dynamic hedging policy and its use of credit lines is possible.

In terms of methodology, our paper is related to Decamps, Mariotti, Rochet, and Villeneuve (2006), who explore a continuous-time model of a firm facing external financing costs. Unlike our set-up, their firm only has a single infinitely-lived project of fixed size, so that they cannot consider the interaction of the firm's real and financial policies. Our model also relates to DeMarzo, Fishman, He, and Wang (2009) who integrate dynamic moral hazard with the q theory of investment (à la Hayashi (1982)) in a continuous-time dynamic optimal contracting framework. Dynamic agency conflicts generate an endogenous financial constraint and induce underinvestment in their model.

[7]Recently, Gamba and Triantis (2008) have extended Hennessy and Whited (2007) to introduce issuance costs of debt and hence obtain the simultaneous existence of debt and cash.

[8]In a related study, DeAngelo, DeAngelo, and Whited (2009) model debt as a transitory financing vehicle to meet the funding needs associated with random shocks to investment opportunities.

II. Model Setup

We first describe the firm's physical production and investment technology, then introduce the firm's external financing costs and its opportunity cost of holding cash, and finally state firm optimality.

A. Production Technology

The firm employs physical capital for production. The price of capital is normalized to unity. We denote by K and I respectively the level of capital stock and gross investment. As is standard in capital accumulation models, the firm's capital stock K evolves according to:

$$dK_t = (I_t - \delta K_t)\, dt, \quad t \geq 0, \tag{1}$$

where $\delta \geq 0$ is the rate of depreciation.

The firm's operating revenue at time t is proportional to its capital stock K_t, and is given by $K_t dA_t$, where dA_t is the firm's revenue or productivity shock over time increment dt. We assume that after accounting for systematic risk the firm's cumulative productivity evolves according to:

$$dA_t = \mu dt + \sigma dZ_t, \quad t \geq 0, \tag{2}$$

where Z is a standard Brownian motion under the risk-neutral measure.[9] Thus, productivity shocks are assumed to be *i.i.d.*, and the parameters $\mu > 0$ and $\sigma > 0$ are the mean and volatility of the risk-adjusted productivity shock dA_t. This production specification is often refereed to as the "AK" technology in the macroeconomics literature.[10] Our assumption of the productivity shocks implies that investment opportunities are constant over time. We intentionally choose such an environment in order to highlight the dynamic effects of financing frictions, not changing

[9] As is standard in asset pricing, we assume that the economy is characterized by a stochastic discount factor Λ_t, which follows $\frac{d\Lambda_t}{\Lambda_t} = -rdt - \eta d\widehat{B}_t$, where $\widehat{B}_t$ is a standard Brownian motion under the physical measure $\mathbb{P}$, and η is the market price of risk (the Sharpe ratio of the market portfolio in the CAPM). Then, $B_t = \widehat{B}_t + \eta t$ will be a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. Finally, Z_t is a standard Brownian motion under $\mathbb{Q}$, and the correlation between Z_t and B_t is ρ. Then, the mean productivity shock under $\mathbb{P}$ is $\hat{\mu} = \mu + \eta\rho\sigma$.

[10] Cox, Ingersoll, and Ross (1985) develop an equilibrium production economy with the "AK" technology. See Jones and Manuelli (2005) for a recent survey in macro.

investment opportunities, on investment, cash, external financing, and risk management policies.

The firm's incremental operating profit dY_t over time increment dt is then given by:

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t)dt, \quad t \geq 0, \tag{3}$$

where I is the gross investment and $G(I, K)$ is the additional adjustment cost that the firm incurs in the investment process. We may interpret dY_t as cash flows from operations. Following the neoclassical investment literature (Hayashi (1982)), we assume that the firm's adjustment cost is homogeneous of degree one in I and K. In other words, the adjustment cost takes the homogeneous form $G(I, K) = g(i)K$, where i is the firm's investment capital ratio ($i = I/K$), and $g(i)$ is an increasing and convex function. Our analyses do not depend on the specific functional form of $g(i)$, and to simplify we assume that $g(i)$ is quadratic:

$$g(i) = \frac{\theta i^2}{2}, \tag{4}$$

where the parameter θ measures the degree of the adjustment cost. Finally, we assume that the firm can liquidate its assets at any time. The liquidation value L_t is proportional to the firm's capital, $L_t = lK_t$, where $l > 0$ is a constant.

The homogeneity assumption embedded in the adjustment cost, the "AK" production technology, and the liquidation technology allows us to deliver our key results in a parsimonious and analytically tractable way. Adjustment costs may not always be convex and the production technology may exhibit decreasing returns to scale in practice, but these functional forms substantially complicate the analysis and do not permit a closed-form characterization of investment and financing policies. As will become clear below, the homogeneity assumption helps reduce the problem to one with effectively a single state variable, which is easier to solve. See Eberly, Rebelo, and Vincent (2008) for empirical evidence in support of the Hayashi homogeneity assumption for the upper quartile of Compustat firms.

B. Information, Incentives and Financing Costs

Neoclassical investment models (à la Hayashi (1982)) assume that the firm faces frictionless capital markets and that the Modigliani and Miller (1958) theorem holds. However, in reality, firms often face important external financing costs due to asymmetric information and managerial incentive problems. Following the classic writings of Jensen and Meckling (1976), Leland and Pyle (1977), and Myers and Majluf (1984) a large empirical literature has sought to measure these costs. For example, Asquith and Mullins (1986) found that the average stock price reaction to the announcement of a common stock issue was -3% and the loss in equity value as a percentage of the size of the new equity issue was -31%.[11] Also, Calomiris and Himmelberg (1997) have estimated the direct *transactions costs* firms face when they issue equity. These are also substantial. In their sample the mean transactions costs, which include underwriting, management, legal, auditing and registration fees as well as the firm's selling concession, are 9% of an issue for seasoned public offerings and 15.1% for initial public offerings.

We do not explicitly model information asymmetries and incentive problems. Rather, to be able to work with a model that can be calibrated we directly model the costs arising from information and incentive problems in reduced form. Thus in our model, whenever the firm chooses to issue external equity we summarize the information, incentive, and transactions costs it then incurs by a fixed cost Φ and a marginal cost γ. Importantly, when firms face fixed costs in raising external equity they will optimally tap equity markets only intermittently and when they do they raise funds in lumps, consistent with observed firm behavior.

To preserve the homogeneity of degree one of our model, we further assume that the firm's fixed cost of issuing external equity is proportional to K, so that $\Phi = \phi K$. Although in practice external costs of financing as a proportion of firm size are more plausibly decreasing with the size of the firm, there are conceptual, mathematical, and economic reasons for modeling these costs as proportional to the size of the firm's capital stock. First, to preserve stationarity it is natural to model costs as

[11] In two related studies based on different data, Masulis and Korwar (1986) and Mikkelson and Partch (1986) found that the average stock price announcement effect of a common stock issue was respectively -3.25% and -4.46%, and the average loss in equity value as a percentage of issue size was respectively -22% and -29.5%.

proportional to K for otherwise the firm would simply *grow out* of its fixed costs.[12] Second, this assumption allows us to keep the model tractable, and generates stationary dynamics for the firm's cash-capital ratio, which are empirically plausible. Third, the information and incentive costs of external financing may to some extent be proportional to the size of the firm. Indeed, the negative announcement effect of a new equity issue affects the firm's entire capitalization. Similarly, the negative incentive effect of a more diluted ownership may also have costs that are proportional to the size of the firm. Note finally, that when we calibrate the model to reflect the circumstances of a given firm in practice we can choose the fixed cost parameter ϕ so that the average issuance cost for that firm is in the ballpark range of the costs the firm is likely to face in reality. Thus, one could apply the model by taking lower values of ϕ for larger firms. The model would then fit the circumstances faced by the firm, at least to a first approximation, even if fixed external financing costs as a proportion of firm size happen to be decreasing in the size of the firm.

We denote by H_t the firm's cumulative external financing up to time t and hence by dH_t the firm's incremental external financing over time interval $(t, t + dt)$. Similarly, we let X_t denote the cumulative costs of external financing up to time t, and dX_t the incremental costs of raising incremental external funds dH_t. The cumulative external equity issuance H and the associated cumulative costs X are stochastic controls chosen by the firm. In the baseline model of this section, external financing is equity.

We now turn to the firm's cash inventory. Let W denote the firm's cash inventory. In our baseline model with no debt, provided that the firm's cash is positive, the firm survives with probability one. However, when the firm runs out of cash ($W_t = 0$) and has no option to borrow, it has to either raise external funds to continue operating, or it must liquidate its assets.[13] If the firm chooses to raise external funds, it must pay the financing costs specified above. In some situations the firm may prefer liquidation, e.g. when the cost of financing is too high relative to the continuation value, or when μ is sufficiently small. Let τ denote the firm's (stochastic) liquidation time. If $\tau = \infty$, then the firm never chooses to liquidate.

[12] Indeed, this is a common assumption in the investment literature. See for example Cooper and Haltiwanger (2006) and Riddick and Whited (2009), among others.

[13] We generalize this specification in Section VIII by allowing the firm to draw on a credit line.

The rate of return that the firm earns on its cash inventory is the risk-free rate r minus a *carry cost* $\lambda > 0$ that captures in a simple way the agency costs that may be associated with *free cash* in the firm.[14] In the presence of such a cost of holding cash, shareholder value is increased when the firm distributes cash back to shareholders should its cash inventory grows too large.[15] Alternatively, the cost of carrying cash may arise from tax distortions. Cash retentions are tax disadvantaged because the associated tax rates generally exceed those on interest income (Graham (2000) and Faulkender and Wang (2006)).

We denote by U the firm's cumulative (non-decreasing) payout to shareholders, and by dU_t the incremental payout over time interval dt. Distributing cash to shareholders may take the form of a special dividend or a share repurchase.[16] The benefit of a payout is that shareholders can invest at the risk-free rate r, which is higher than $(r - \lambda)$ the net rate of return on cash within the firm. However, paying out cash also reduces the firm's cash balance, which potentially exposes the firm to current and future under-investment and future external financing costs.

Combining cash flow from operations dY_t given in (3), with the firm's financing policy given by the cumulative payout process U and the cumulative external financing process H, the firm's cash inventory W evolves according to the following cash-accumulation equation:

$$dW_t = dY_t + (r - \lambda)\ W_t dt + dH_t - dU_t, \tag{5}$$

where the second term is the interest income (net of the carry cost λ), the third term dH_t is the cash inflow from external financing, and the last term dU_t is the cash outflow to investors, so that $(dH_t - dU_t)$ is the net cash flow from financing. This equation is a general accounting identity, where dH_t, dU_t, and dY_t are endogenously determined by the firm.

The firm's financing opportunities is time-invariant in our model, which is not realistic. However,

[14] This assumption is standard in models with cash. See e.g. Kim, Mauer, and Sherman (1998) and Riddick and Whited (2008).

[15] If $\lambda = 0$, the firm has no reason to pay out cash since keeping cash inside the firm has no costs, but still has the benefits of relaxing financing constraints. Another possibility is $\lambda < 0$. If the firm is better at identifying investment opportunities than investors, $-\lambda$ can be treated as an excess return. We do not explore this case in this paper.

[16] A commitment to regular dividend payments is suboptimal in our model. We exclude any fixed or variable payout costs, which can be added to the analysis.

we choose constant investment and financing opportunities so as to highlight the impact of financing frictions on investment, firm value, and risk management policies without appealing to arguments such as market timing incentives induced by time-varying financing opportunities. As we will show, despite our stylized assumptions, the interaction of fixed/proportional financing costs with real investment generate several novel and economically significant insights.

Firm optimality. The firm chooses its investment I, cumulative payout policy U, cumulative external financing H, and liquidation time τ to maximize shareholder value defined below:

$$\mathbb{E}\left[\int_0^\tau e^{-rt}\left(dU_t - dH_t - dX_t\right) + e^{-r\tau}\left(lK_\tau + W_\tau\right)\right]. \tag{6}$$

The expectation is taken under the risk-adjusted probability. The first term is the discounted value of net payouts to shareholders and the second term is the discounted value upon liquidation. Optimality may imply that the firm never liquidates. In that case, we have $\tau = \infty$. We impose the usual regularity conditions to ensure that the optimization problem is well posed. Our optimization problem is most obviously seen as characterizing the benchmark for the firm's efficient investment, cash-inventory, dynamic hedging, payout, and external financing policy when the firm faces external financing and cash carry costs. However, as in the dynamic investment literature with financial frictions, this formulation can also be viewed as representing a principal-agent problem with reduced-form financial frictions.[17] The main advantage of this short-cut is that we are able to work with a much more tractable dynamic framework, which in particular easily lends itself to calibrations. It is clearly desirable to push the analysis further and to explicitly model the agent's objective function and incentive constraints.[18]

[17] The key simplification relative to a classic principal-agent setup is that we only model agency costs (that is, the costs of structuring the agent's compensation to align her interests with those of shareholders) in reduced form. A natural way of interpreting these costs is as monitoring costs to ensure that the agent acts in the interest of shareholders.

[18] For a model of dynamic incentive problem in a q theory of investment framework, see DeMarzo, Fishman, He, and Wang (2009).

III. The Neoclassical Benchmark

We first summarize the solution for the neoclassical q theory of investment, in which the Modigliani-Miller theorem holds. The firm's first-best investment policy is given by $I^{FB} = i^{FB}K$, where[19]

$$i^{FB} = r + \delta - \sqrt{(r+\delta)^2 - 2(\mu - (r+\delta))/\theta}. \tag{7}$$

The value of the firm's capital stock is $q^{FB}K$, where q^{FB} is Tobin' s q given by:

$$q^{FB} = 1 + \theta i^{FB}. \tag{8}$$

Two observations are in order. First, due to the homogeneity property in production , marginal q is equal to average (Tobin's) q, as in Hayashi (1982). Second, gross investment I is positive if and only if the expected productivity μ is higher than $r + \delta$. With $\mu > r + \delta$ and hence positive investment, installed capital earns rents. Therefore, Tobin's q is greater than unity due to adjustment costs. Next, we analyze the problem of a financially constrained firm.

IV. Model Solution

When the firm faces costs of raising external funds, it can reduce future financing costs by retaining earnings (i.e. hoarding cash) to finance its future investments. Firm value then depends on two natural state variables, its stock of cash W and its capital stock K. Let $P(K,W)$ denote the firm value. We show that firm decision-making and firm value then depend on which of the following three regions it finds itself in: i) an external funding/liquidation region, ii) an internal financing region, and iii) a payout region. As will become clear below, the firm is in the external funding/liquidation region when its cash stock W is less than or equal an endogenous *lower barrier* $\underline{W}$. It is in the payout region when its cash stock W is greater than or equal an endogenous *upper barrier* $\overline{W}$. And it is in the internal financing region when $W \in (\underline{W}, \overline{W})$. We first characterize the

[19]To ensure that the first-best investment policy is well defined, the following parameter restriction has to be imposed: $(r+\delta)^2 - 2(\mu - (r+\delta))/\theta > 0$.

solution in the internal financing region.

A. Internal Financing Region

In this region, firm value $P(K,W)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rP(K,W) = \max_I \ (I - \delta K)\, P_K + [(r-\lambda)W + \mu K - I - G(I,K)]P_W + \frac{\sigma^2 K^2}{2} P_{WW}. \tag{9}$$

The first term (the P_K term) on the right side of (9) represents the marginal effect of net investment $(I - \delta K)$ on firm value $P(K,W)$. The second term (the P_W term) represents the effect of the firm's expected saving on firm value, and the last term (the P_{WW} term) captures the effects of the volatility of cash holdings W on firm value.

The firm finances its investment out of the cash inventory in this region. The convexity of the physical adjustment cost implies that the investment decision in our model admits an interior solution. The investment-capital ratio $i = I/K$ then satisfies the following first-order condition:

$$1 + \theta i = \frac{P_K(K,W)}{P_W(K,W)}. \tag{10}$$

With frictionless capital markets (the MM world) the marginal value of cash is $P_W = 1$, so that the neoclassical investment formula obtains: $P_K(K,W)$ is the marginal q, which at the optimum is equal to the marginal cost of adjusting the capital stock $1 + \theta i$. With costly external financing, on the other hand, the investment Euler equation (10) captures both real and financial frictions. The marginal cost of adjusting physical capital $(1 + \theta i)$ is now equal to the ratio of marginal q, $P_K(K,W)$, to the marginal cost of financing (or equivalently, the marginal value of cash), $P_W(K,W)$. Thus, the more costly the external financing (the higher P_W) the less the firm invests, *ceteris paribus*.

A key simplification in our setup is that the firm's two-state optimization problem can be reduced to a one-state problem by exploiting homogeneity. That is, we can write firm value as

$$P(K,W) = K \cdot p(w), \tag{11}$$

where $w = W/K$ is the firm's cash-capital ratio, and reduce the firm's optimization problem to a one-state problem in w. The dynamics of w can be written as:

$$dw_t = (r - \lambda)w_t dt - (i(w_t) + g(i(w_t))dt + (\mu dt + \sigma dZ_t). \tag{12}$$

The first term on the right-hand side is the interest income net of cash-carrying costs. The second term is the total flow-cost of (endogenous) investment (capital expenditures plus adjustment costs). While most of the time we have $i(w_t) > 0$, the firm may sometimes want to engage in asset sales (i.e. set $i(w_t) < 0$) in order to replenish its stock of cash and thus delay incurring external financing costs. Finally, the third term is the realized revenue per unit of capital (dA). In accounting terms, this equation provides the link between the firm's income statement (source and use of funds) and its balance sheet.

Instead of solving for firm value $P(K, W)$, we only need to solve for the firm's value-capital ratio $p(w)$. Note that marginal q is $P_K(K, W) = p(w) - wp'(w)$, the marginal value of cash is $P_W(K, W) = p'(w)$, and $P_{WW} = p''(w)/K$. Substituting these terms into (9) we obtain the following ordinary differential equation (ODE) for $p(w)$:

$$rp(w) = (i(w) - \delta)\left(p(w) - wp'(w)\right) + ((r - \lambda)w + \mu - i(w) - g(i(w)))\, p'(w) + \frac{\sigma^2}{2} p''(w)\,. \tag{13}$$

We can also simplify the FOC (10) to obtain the following equation for the investment-capital ratio $i(w)$:

$$i(w) = \frac{1}{\theta}\left(\frac{p(w)}{p'(w)} - w - 1\right). \tag{14}$$

Using the solution $p(w)$ and substituting for this expression of $i(w)$ in (12) we thus obtain the equation for the firm's optimal accumulation of w.

To completely characterize the solution for $p(w)$, we must also determine the boundaries $\underline{w}$ at which the firm raises new external funds (or closes down), how much to raise (the target cash-capital ratio after issuance), and $\overline{w}$ at which the firm pays out cash to shareholders.

B. Payout Region

Intuitively, when the cash-capital ratio is very high, the firm is better off paying out the excess cash to shareholders to avoid the carry carry cost. The natural question is how high the the cash-capital ratio needs to be before the firm pays out. Let $\overline{w}$ denote this endogenous payout boundary. Intuitively, if the firm starts with a large amount of cash ($w > \overline{w}$), then it is optimal for the firm to distribute the excess cash as a lump-sum and bring the cash-capital ratio w down to $\overline{w}$. Moreover, firm value must be continuous before and after cash distribution. Therefore, for $w > \overline{w}$, we have the following equation for $p(w)$:

$$p(w) = p(\overline{w}) + (w - \overline{w}), \quad w > \overline{w}. \tag{15}$$

Since the above equation also holds for w close to $\overline{w}$, we may take the limit and obtain the following condition for the endogenous upper boundary $\overline{w}$:

$$p'(\overline{w}) = 1. \tag{16}$$

At $\overline{w}$ the firm is indifferent between distributing and retaining one dollar, so that the marginal value of cash must equal one, which is the marginal cost of cash to shareholders. Since the payout boundary $\overline{w}$ is optimally chosen, we also have the following "super contact" condition (see, e.g. Dumas (1991)):

$$p''(\overline{w}) = 0. \tag{17}$$

C. External Funding/Liquidation Region

When the firm's cash-capital ratio w is less than or equal to the lower barrier $\underline{w}$, the firm either incurs financing costs to raise new funds or liquidates. Depending on parameter values, it may prefer either liquidation or refinancing by issuing new equity. Although the firm can choose to liquidate or raise external funds at any time, we show that it is optimal for the firm to wait until it runs out of cash, i.e. $\underline{w} = 0$. The intuition is as follows. First, because investment incurs convex adjustment cost and the production is an efficient technology (in the absence of financing costs), the firm does not want to prematurely liquidate. Second, in the case of external financing, cash within the firm earns a below-market interest rate $(r - \lambda)$, while there is also time value for the external financing costs. Since investment is smooth (due to convex adjustment cost), the firm can always pay for any level of investment it desires with internal cash as long as $w > 0$. Thus, without any benefit for early issuance, it is always better to defer external financing as long as possible. The above argument highlights the robustness of the pecking order between cash and external financing in our model. With stochastic financing cost or stochastic arrival of growth options, the firm may time the market by raising cash in times when financing costs are low. See Bolton, Chen, and Wang (2009).

When the expected productivity μ is low and/or cost of financing is high, the firm will prefer liquidation to refinancing. In that case, because the optimal liquidation boundary is $\underline{w} = 0$, firm value upon liquidation is thus $p(0)K = lK$. Therefore, we have

$$p(0) = l. \tag{18}$$

If the firm's expected productivity μ is high and/or its cost of external financing is low, then it is better off raising costly external financing than liquidating its assets when it runs out of cash. To economize fixed issuance costs $(\phi > 0)$, firms issue equity in lumps. With homogeneity, we can show that total equity issue amount is mK, where $m > 0$ is endogenously determined as follows. First, firm value is continuous before and after equity issuance, which implies the following condition for $p(w)$ at the boundary $\underline{w} = 0$:

$$p(0) = p(m) - \phi - (1 + \gamma)\, m. \tag{19}$$

The right side represents the firm value-capital ratio $p(m)$ minus both the fixed and the proportional costs of equity issuance, per unit of capital. Second, since m is optimally chosen, the marginal value of the last dollar raised must equal one plus the marginal cost of external financing, $1 + \gamma$. This gives the following smoothing pasting boundary condition at m:

$$p'(m) = 1 + \gamma. \tag{20}$$

D. Piecing the Three Regions Together

To summarize, for the liquidation case, the complete solution for the firm's value-capital ratio $p(w)$ and its optimal dynamic investment policy is given by: i) the HJB equation (13); ii) the investment-capital ratio equation (14), and; iii) the liquidation (18) and payout boundary conditions (16)-(17).

Similarly, when it is optimal for the firm to refinance rather than liquidate, the complete solution for the firm's value-capital ratio $p(w)$ and its optimal dynamic investment and financing policy is given by: i) the HJB equation (13); ii) the investment-capital ratio equation (14); iii) the equity-issuance boundary condition (19); iv) the optimality condition for equity issuance (20), and; v) the endogenous payout boundary conditions (16)-(17). Finally, to verify that refinancing is indeed the firm's global optimal solution, it is sufficient to check that $p(0) > l$.

V. Quantitative Analysis

We now turn to quantitative analysis of our model. For the benchmark case, we set the riskfree rate at $r = 6\%$ and adopt the following technological parameter values. The rate of depreciation is $\delta = 10\%$. The mean and volatility of the risk-adjusted productivity shock are $\mu = 18\%$ and $\sigma = 9\%$, respectively, which are in line with the estimates of Eberly, Rebelo, and Vincent (2008) for large US firms. These parameters are all annualized. The adjustment cost parameter is $\theta = 1.5$ (see Whited (1992)). The implied first-best q in the neoclassical model is then $q^{FB} = 1.23$, and the corresponding first-best investment-capital ratio is $i^{FB} = 15.1\%$. We then set the cash-carrying cost parameter to $\lambda = 1\%$. The proportional financing cost is $\gamma = 6\%$ (as suggested in Sufi (2009)) and the fixed cost of financing is $\phi = 1\%$, which jointly generate average equity financing costs that are consistent with the data. Finally, for the liquidation value we take $l = 0.9$ (as suggested in Hennessy and Whited (2007)).

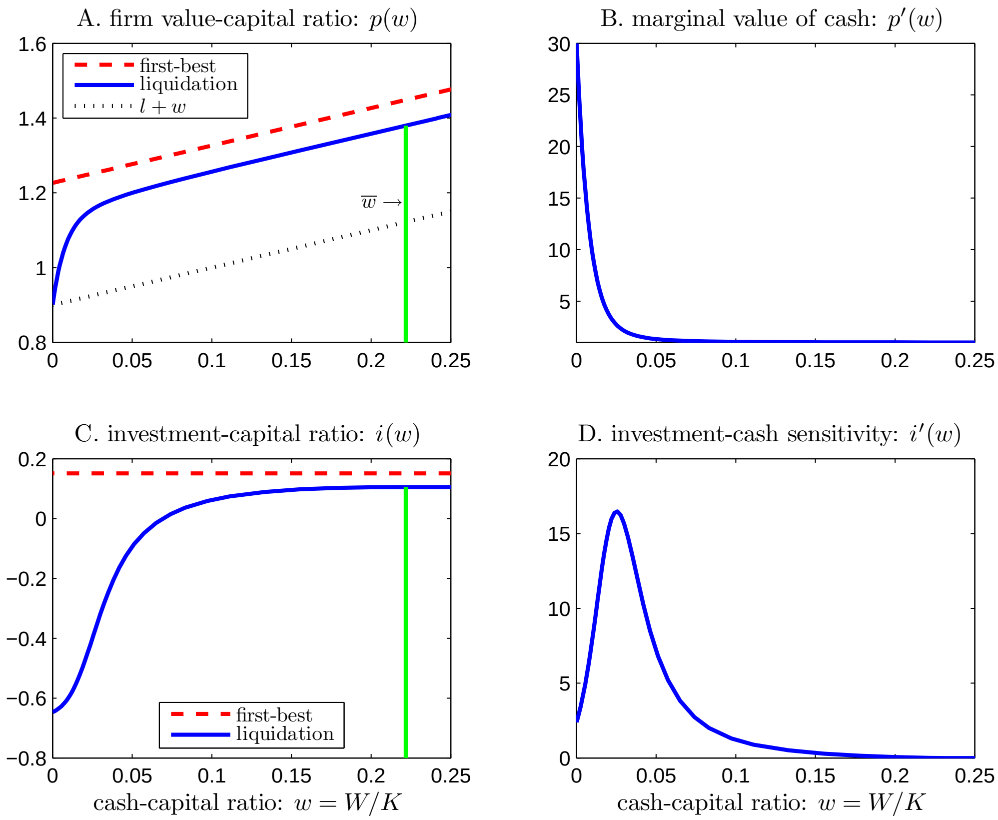

Figure 1: **Case I. Liquidation.** This figure plots the solution in the case when the firm has to liquidate when it runs out of cash ($w = 0$). The parameters are: riskfree rate $r = 6\%$, the mean and volatility of increment in productivity $\mu = 18\%$ and $\sigma = 9\%$, adjustment cost parameter $\theta = 1.5$, capital depreciation rate $\delta = 10\%$, cash-carrying cost $\lambda = 1\%$, and liquidation value-capital ratio $l = 0.9$.

Before analyzing the impact of external equity financing, for comparison, we first consider a special case where the firm is forced to liquidate when it runs out of cash.

Case I: Liquidation. Figure 1 plots the solution in the liquidation case. In Panel A, the firm's value-capital ratio $p(w)$ starts at $l = 0.9$ (liquidation value) when cash balances are equal to 0, is concave in the region between 0 and the endogenous payout boundary $\overline{w} = 0.22$, and becomes linear (with slope 1) beyond the payout boundary ($w \geq \overline{w}$). In Section IV, we have argued that the firm will never liquidate before its cash balances hit 0. Panel A of Figure 1 provides a graphic illustration of this result, where $p\,(w)$ lies above the liquidation value $l + w$ (normalized by capital) for all $w > 0$.

Panel B of Figure 1 plots the marginal value of cash $p'(w) = P_W(K, W)$. The marginal value of cash increases as the firm becomes more constrained and liquidation becomes more likley. It also shows that the firm value is concave in the internal financing region ($p''(w) < 0$). The external financing constraint makes the firm hoard cash today in order to reduce the likelihood that it will be liquidated in the future, which effectively induces "risk aversion" for the firm. Consider the effect of a mean-preserving spread of cash holdings on the firm's investment policy. Intuitively, the marginal cost from a smaller cash holding is higher than the marginal benefit from a larger cash holding because the increase in the likelihood of liquidation outweighs the benefit from otherwise relaxing the firm's financial constraints. It is the concavity of the value function that gives rise to the demand for risk management. Observe also that the marginal value of cash reaches a staggering value of 30 as w approaches 0. In other words, an extra dollar of cash is worth as much as \$30 to the firm in this region, because it helps keep the firm away from costly liquidation.

Panel C plots the investment-capital ratio $i(w)$ and illustrates under-investment due to the extreme external financing constraints. Optimal investment by a financially constrained firm is always lower than first-best investment $i^{FB} = 15.1\%$, but especially when the firm's cash inventory w is low. Actually, when w is sufficiently low the firm will disinvest by selling assets to raise cash and move away from the liquidation boundary. Note that disinvestment is costly not only because the firm is underinvesting but also because it incurs physical adjustment costs when lowering its capital stock. For the parameter values we use, asset sales (disinvestments) are at the annual rate of over 60% of the capital stock when w is close to zero! The firm tries very hard not to be forced into liquidation, which would permanently eliminate the firm's future growth opportunities. Note also that even at the payout boundary, the investment-capital ratio is only $i(\overline{w}) = 10.6\%$, about 30% lower than the first best level i^{FB}. Intuitively, the firm is trading off the cash-carrying costs with the cost of underinvestment. It will optimally choose to hoard more cash and invest more at the payout boundary when the cash-carrying cost λ is lower.

Next, we consider a measure of the investment-cash sensitivity given by $i'(w)$.[20] Taking the

[20] The notion of "investment-cash sensitivity" we define here should be interpreted with caution empirically. While $i'(w)$ measures how investment changes in response to *exogenous* shocks to cash holding in the model, the changes in cash we observe empirically are likely to be correlated with changes in investment opportunities and financing costs. The same point also applies to the interpretation of "marginal value of cash" $p'(w)$.

derivative of investment-capital ratio $i(w)$ in (14) with respect to w, we get

$$i'(w) = -\frac{1}{\theta}\frac{p(w)p''(w)}{p'(w)^2} > 0. \tag{21}$$

The concavity of p ensures that $i'(w) > 0$ in the internal financing region, which is confirmed in Panel D of Figure 1. Remarkably, the investment-cash sensitivity $i'(w)$ is not monotonic in w. In particular, when the cash holding is sufficiently low, $i'(w)$ actually increases with the cash-capital ratio. Formally, the slope of $i'(w)$ depends on the third derivative of $p(w)$, for which we do not have an analytical characterization.

Clearly, liquidation is very inefficient in our model (recall that the marginal value of cash at liquidation is 30 and asset sale is at an annual rate of 60%). Next we consider the more realistic setting where the firm is allowed to issue equity provided it pays the financing costs.

Case II: Refinancing. Figure 2 displays the solutions for both the case with fixed financing costs ($\phi = 1\%$) and without ($\phi = 0$). Observe that at the financing boundary $\underline{w} = 0$, the firm's value-capital ratio $p(w)$ is strictly higher than l, so that external equity financing is preferred to liquidation in equilibrium. Comparing with the liquidation case, we find that the endogenous payout boundary (marked by the solid vertical line on the right) is $\overline{w} = 0.19$ when $\phi = 1\%$, lower than the payout boundary for the case where the firm is liquidated ($\overline{w} = 0.22$). Not surprisingly, firms are more willing to pay out cash when they can raise new funds in the future. The firm's optimal *return cash-capital ratio* for our parameter values is $m = 0.06$, and is marked by the vertical line on the left in Panel A. Without fixed cost ($\phi = 0$), the payout boundary drops to $\overline{w} = 0.14$, substantially lower than the ones with the fixed costs and the liquidation case. In this case, the firm's return cash-capital ratio is zero. In other words, the firm raises just enough funds to keep w above 0. This is consistent with the intuition that the higher the fixed cost parameter ϕ, the bigger the size of refinancing (higher *return cash-capital ratio* m) each time the firm raises cash.

Panel B plots the marginal value of cash $p'(w)$, which is positive and decreasing, confirming that $p(w)$ is strictly concave for $w \leq \overline{w}$. Conditional on issuing equity and having paid the fixed financing cost, the firm optimally chooses the return cash-capital ratio m such that the marginal

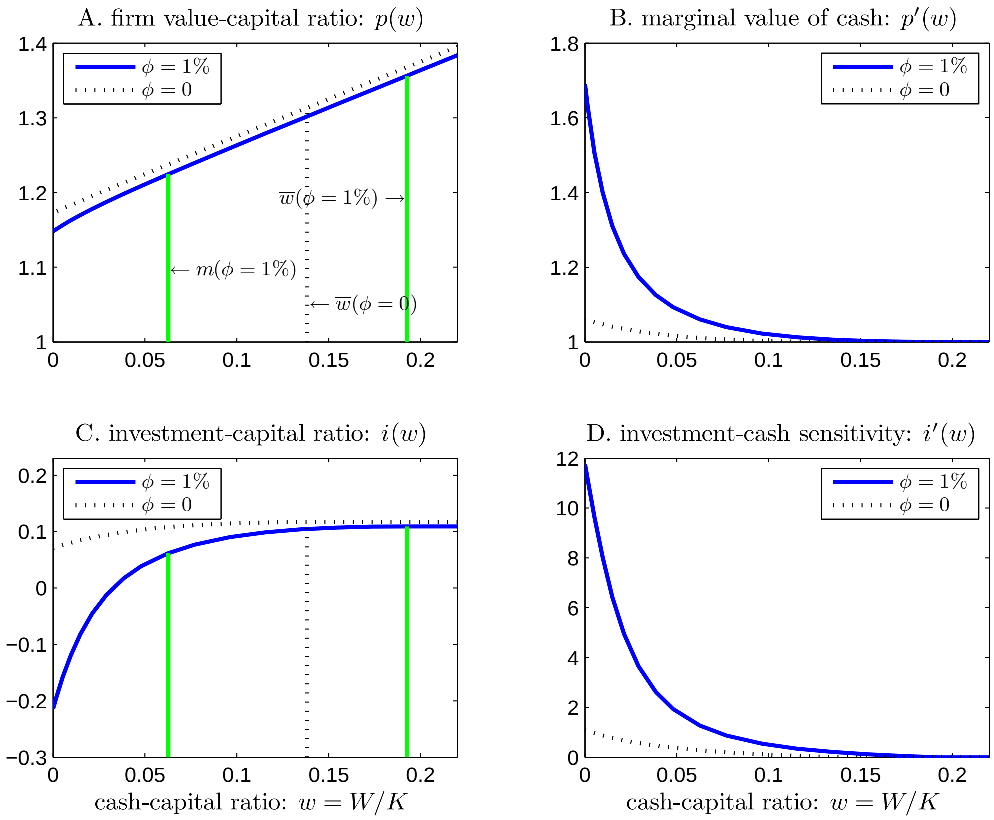

Figure 2: **Case II. Optimal refinancing at $w = 0$.** This figure plots the solution in the case of refinancing. The parameters are: riskfree rate $r = 6\%$, the mean and volatility of increment in productivity $\mu = 18\%$ and $\sigma = 9\%$, adjustment cost parameter $\theta = 1.5$, capital depreciation rate $\delta = 10\%$, cash-carrying cost $\lambda = 1\%$, proportional and fixed financing costs $\gamma = 6\%$, $\phi = 1\%$.

value of cash $p'(m)$ is equal to the marginal cost of issuance $1 + \gamma$. To the left of the return cash-capital ratio m, the marginal value of cash $p'(w)$ lies above $1 + \gamma$, reflecting the fact that the fixed cost component in raising equity increases the marginal value of cash. When the firm runs out of cash, the marginal value of cash is around 1.7, much higher than $1 + \gamma = 1.06$. This result highlights the importance of fixed financing costs: even a moderate fixed cost can substantially raise the marginal value of cash in the low-cash region.

As in the previous case, the investment-capital ratio $i(w)$ is increasing in w. It reaches the peak at the payout boundary $\overline{w}$, where $i(w) = 11\%$. Higher fixed cost component effectively increases the severity of financing constraints, therefore leading to more underinvestment. This is particularly true in the region to the left of the return cash-capital ratio m, where the investment-capital ratio

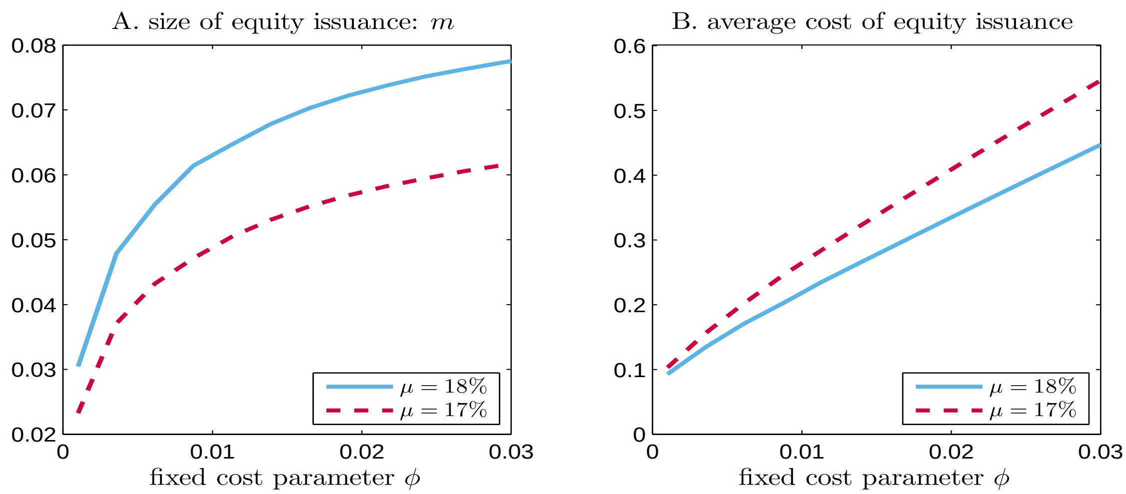

Figure 3: **Relative size and average cost of equity issuance.** This figure plots the size of equity issuance relative to capital (m) and the average cost of equity issuance (AC) for different levels of fixed cost of issuance and expected productivity.

$i(w)$ drops rapidly and even involves asset sales (about 20% of total capital when w approaches 0). Asset sales go down quickly ($i'(w) > 10$) when w is close to zero. This is because both asset sales and equity issuance are very costly. In contrast, removing the fixed financing costs greatly alleviates the under-investment problem, and the investment-capital ratio $i(w)$ becomes essentially flat except for very low w.

Next, we briefly consider how the optimal size of equity issues m varies with the financing cost parameters ϕ and the firm's expected productivity μ. Intuitively, m should be increasing in ϕ, as the firm seeks to lower its average cost of external funds by increasing the size of its issue when ϕ is higher. Moreover, one expects m to be concave in ϕ as the marginal value of cash $p'(w)$ is decreasing in w. Both features are confirmed numerically in Panel A of Figure 3.

In reality neither financing cost parameters (γ, ϕ) nor expected productivity μ are easy to observe. Empirical studies estimating external financing costs have focused on the average cost of external financing, defined as the ratio of total financing costs and the size of the equity issue m:

$$AC = \frac{\phi}{m} + \gamma.$$

The fixed cost parameter ϕ of equity issuance is often perceived to be larger for smaller firms, and therefore one would expect to see these firms to have higher average costs, other things equal. However, smaller firms are also likely to have higher μ. This will raise the optimal size of their equity issues m as is highlighted in Panel A. Therefore, the relation between average issuance costs and firm size is ambiguous. Panel B of Figure 3 demonstrates this observation.

This discussion highlights the importance of heterogeneity and endogeneity issues when measuring issuance costs. It helps explain why there may not be a clear relation between firm size and average costs of issues in the data, and sheds light on the empirical debate over the nature of scale economies in equity issuance and whether equity issuance costs are primarily fixed or variable (see Lee, Lochhead, Ritter, and Zhao (1996), Calomiris and Himmelberg (1997) and Calomiris, Himmelberg, and Wachtel (1995)).

Average q, marginal q, and investment. We now turn to the model's predictions on average and marginal q. We take the firm's *enterprise value* – the value of all the firm's marketable claims minus cash, $P(K,W) - W$ – as our measure of the value of the firm's capital stock. Average q, denoted by $q_a(w)$, is then the firm's enterprise value divided by its capital stock:

$$q_a(w) = \frac{P(K,W) - W}{K} = p(w) - w. \tag{22}$$

First, note that average q already increase with w. This can be seen from $q_a'(w) = p'(w) - 1 \geq 0$, where the inequality follows from the fact that marginal value of cash is weakly greater than unity. Second, average q is concave provided that $p(w)$ is concave, in that $q_a''(w) = p''(w)$.

In our model where external financing is costly, marginal q, denoted by $q_m(w)$, is given by

$$q_m(w) = \frac{d\left(P(K,W) - W\right)}{dK} = p(w) - wp'(w) = (p(w) - w) - \big(p'(w) - 1\big)\, w. \tag{23}$$

Recall that in the neoclassical setting (Hayashi (1982)), average q equals marginal q. In our model, average q differs from marginal q due to the external financing costs. An increase in the capital stock K has two effects on the firm's enterprise value. The first effect is captured by the

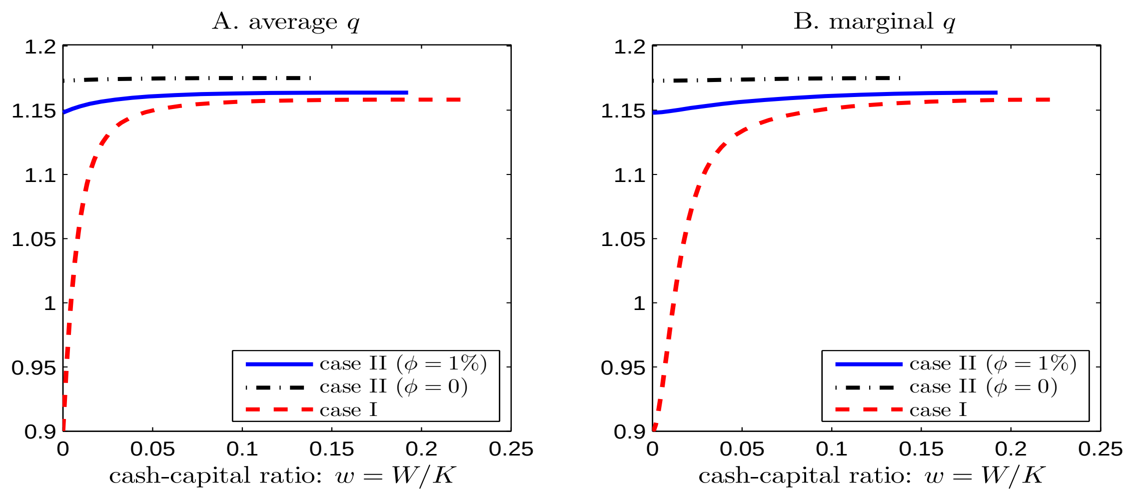

Figure 4: **Average q and marginal q.** This figure plots the average q and marginal q from the three special cases of the model. The right end of each line corresponds to the respective payout boundary, beyond which both q_a and q_a are flat.

term $(p(w) - w)$ and reflects the direct effect of an increase in capital on firm value, holding w fixed. This term is equal to average q. The second term $(p'(w) - 1)\,w$ reflects the effect of external financing costs on firm value through w. Increasing the capital stock mechanically lowers the cash-capital ratio $w = W/K$ for a given cash inventory W. As a result, the firm's financing constraint becomes tighter and firm value drops, *ceteris paribus*.

Figure 4 plots the average and marginal q for the liquidation case, the refinancing case with no fixed costs ($\phi = 0$), and the refinancing cost with fixed costs ($\phi = 1\%$). The average and marginal q are below the first best level, $q^{FB} = 1.23$ in all three cases, and they become lower as external financing becomes more costly. The marginal value of cash $p'(w)$ is always larger than one due to costly external financing. As a result, average q increases with w. Also, the concavity of $p(w)$ implies that marginal q increases with w. From (22) and (23), we see that $p'(w) > 1$ and $w > 0$ imply that $q_m(w) < q_a(w)$, as displayed in Figure 4.

Stationary distributions of w, $p(w)$, $p'(w)$, $i(w)$, average q, and marginal q. We next investigate the stationary distributions for the key variables tied to optimal firm policies in the refinancing case ($\phi = 1\%$). We first simulate the cash-capital ratio under the physical probability

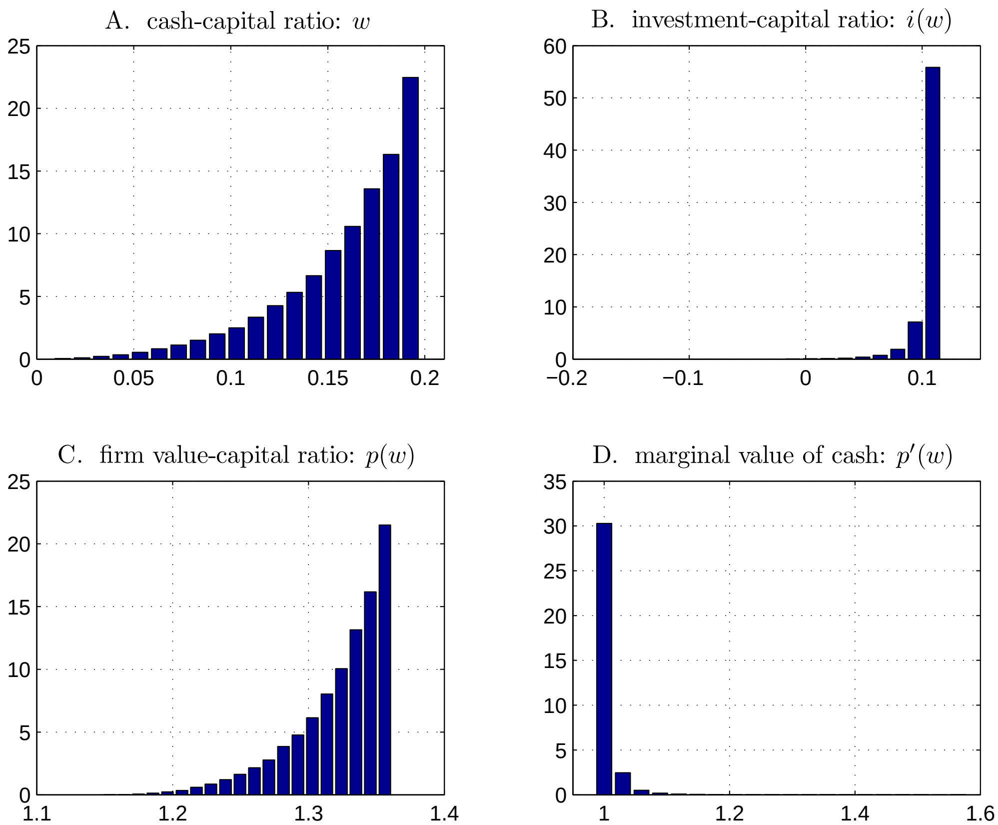

Figure 5: **Stationary distributions in the case of refinancing.** This figure plots the stationary distributions of 4 variables in Case II with $\phi = 1\%$.

measure. To do so, we calibrate the Sharpe ratio of the market portfolio $\eta = 0.3$, and assume that the correlation between the firm technology shocks and the market return is $\rho = 0.8$. Then, the mean of the productivity shock under the physical probability is $\hat{\mu} = 0.20$. Figure 5 shows the distributions for the cash-capital ratio w, the value-capital ratio $p(w)$, the marginal value of cash $p'(w)$, and the investment-capital ratio $i(w)$. Since $p(w), p'(w), i(w)$ are all monotonic in Case II, the densities for their stationary distributions are connected with that of w through (the inverse of) their derivatives.

Strikingly, the cash holdings of a firm are relatively high most of the time, and hence the probability mass for $i(w)$ and $p(w)$ is concentrated at the highest values in the relevant support of w. The marginal value of cash $p'(w)$ is therefore also mostly around unity. Thus, the firm's optimal cash management policies appear to be effective at shielding it from severe financing constraints

Table I: Moments from the stationary distribution of the refinancing case

This table reports the population moments for cash-capital ratio (w), investment-capital ratio ($i(w)$), marginal value of cash ($p'(w)$), average q ($q_a(w)$), and marginal q ($q_m(w)$) from the stationary distribution in Case II ($\phi = 1\%$).

	w	$i(w)$	$p'(w)$	$q_a(w)$	$q_m(w)$
mean	0.159	0.104	1.006	1.164	1.163
median	0.169	0.108	1.001	1.164	1.164
std	0.034	0.013	0.018	0.001	0.001
skewness	-1.289	-6.866	9.333	-8.353	-3.853
kurtosis	4.364	76.026	146.824	106.580	22.949

and underinvestment most of the time.

Table 1 reports the mean, median, standard deviation, skewness, and kurtosis for w, $i(w)$, $p(w)$, $p'(w)$, average q ($q_a(w)$) and marginal q ($q_m(w)$). Not surprisingly, all these variables have skewness. Other than the marginal value of cash $p'(w)$, all remaining five variables have negative skewness with medians larger than the respective means. The positive skewness for $p'(w)$ is consistent with the negative skewness of all the other five variables, as $p'(w)$ is highest for low values of w due to the concavity of $p(w)$. Note also that all these variables have *fat tails*. Interestingly, the kurtosis values for $p'(w)$ and $q_a(w)$ are large, despite their small standard deviations and the small difference between the mean and median values of both $p'(w)$ and $q_a(w)$.

Existing empirical research on corporate cash inventory has mostly focused on firms' average holdings (the first entry in the first row of Table 1) and highlighted that average holdings have increased in recent years. Our model gives a more complete picture of the dynamics of firm capital expenditures and cash holdings. It provides predictions on the time series behavior of firm's investment and financing policies, their valuation, as well as the cross-sectional distribution of cash holdings, and the joint distribution of cash holdings, investment, Tobin' s q, and the frequency of external financing.

As is apparent from Table I, average cash holdings provide an incomplete and even misleading picture of firms' cash management, investment, and valuation. Indeed, one observes that even though the median and the mean of the firm's marginal value of cash $p'(w)$ is close to unity, with a standard deviation of only 0.018, there is a huge kurtosis (146.8) indicating that firms are exposed to potentially large financing costs even if their marginal value of cash is close to unity on average.

Despite the tight distributions for average q and $i(w)$, the mean and median of $q_a(w)$ are 1.16, which is about 5% lower than $q^{FB} = 1.23$, the average q for a firm without external financing costs. Similarly, the mean and median of $i(w)$ is 0.104, which is about 31% lower than $i^{FB} = 0.151$, the investment-capital ratio for a firm without external financing costs. Therefore, simply looking at the difference between the mean and the median or even the standard deviation for these variables, one can end up with a misleading description of firms' financing constraints. The observation that the ratio of the median to mean marginal value of cash $p'(w)$ is close to unity, in particular, does not imply that firm financing constraints are small. The endogeneity of firms' cash holdings mitigates the time-varying impact of financing costs on investment, but the effects remain large on average.

Even under the assumptions of constant investment and financing opportunity sets, we still gain significant economic insights from the simulation exercise. In particular, it shows that firms respond to the financing constraints by optimally managing their cash holdings so as to stay away most of the time from financial distress situations. We note that the distributions simulated from the model are not meant to accurately match the data. The accuracy of the simulated distributions can be substantially improved if we allow for changing investment and financing opportunity sets and firm heterogeneity.

Comparative statics. We close this section with a comparative statics analysis of firm cash holdings and investment for the following six parameters: $\mu, \theta, r, \sigma, \phi, \lambda$. We divide these parameters into two categories. The first three (μ, θ, r) are parameters on the physical side and have direct effects on investment (see i^{FB} in equation(7)); the rest (σ, ϕ, λ) only affect investment and firm value through financing constraints. We examine the effects of these parameters through their impact on the distributions of cash holdings and investment in Figure 6 and 7.

In Figure 6, the left panels (A, C, and E) plot the cumulative stationary distributions (CDF) of the cash holdings w, and the right panels (B, D, and F) plot the cumulative distributions of firm

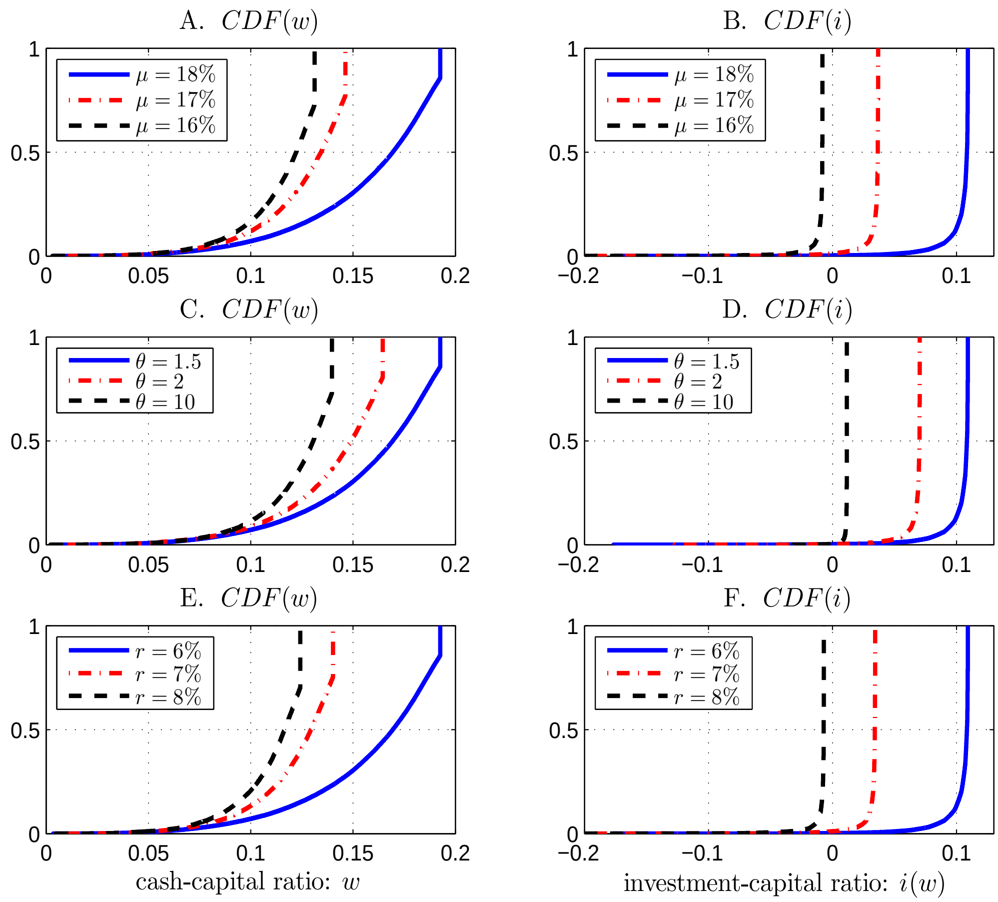

Figure 6: **Comparative statics I: μ, θ, and r.** This figure plots the cumulative distribution function for the stationary distribution of cash-capital ratio (w) and investment-capital ratio ($i(w)$) for different values of the mean of productivity shocks μ, investment adjustment cost θ, and interest rate r.

investments i. As Panel A highlights, when mean productivity increases (from $\mu = 16\%$ to $\mu = 18\%$) firms tend to hold more cash. That is, the cumulative distributions of firms for higher values of μ first-order stochastically dominate the distributions for lower values of μ. This is intuitive, since the return on investment increases with μ so that the shadow value of cash increases. Still, one might expect firms to spend their cash more quickly for higher μ as the value of investment opportunities rises, so that the net effect on firm cash holdings is ambiguous a priori. In our baseline model, the net effect on w of a higher μ is positive, because investment adjustment costs induce firms to only gradually increase their investment outlays in response to an increase in μ.

The effect of an increase in μ on investment is highlighted in Panel B. Firms respond to an increase in μ by increasing investment. For $\mu = 16\%$ firms are disinvesting as $i(w)$ is negative for all firms. For $\mu = 17\%$ nearly all firms are making positive investments, with most firms bunched at an investment level of roughly $i(w) = 3.5\%$. Finally, for $\mu = 18\%$ most firms are investing close to $i(w) = 11\%$.

The effects of an increase in investment adjustment cost θ and interest rate r on cash holdings and investment are also quite intuitive. As Panel D shows, an increase in θ has a negative effect on investment. If firms invest less, one should expect their cash holdings to increase almost mechanically. However, this turns out not to be the case. Firms have a lower shadow value of cash if they anticipate lower future investment outlays. Therefore they end up holding less cash, as is illustrated in Panel C. Similar comparative statics hold for increases in the risk-free rate r: with higher interest rates firms invest less and therefore hold less cash. This is indeed the case, as is illustrated in Panel E and F.

The effects of an increase in the idiosyncratic volatility of productivity shocks are shown in Panels A and B of Figure 7, where the stationary distribution is plotted for values of $\sigma = 7\%$, $\sigma = 9\%$ and $\sigma = 11\%$. We change σ by changing the idiosyncratic volatility while holding the systematic volatility fixed, so that the risk-adjusted mean productivity shock μ is unaffected. Again, it is intuitive that firms respond to greater underlying volatility of productivity shocks by holding more cash. Higher cash reserves, in turn, tend to raise the average cost of investment, so that one might expect a higher σ to induce firms to scale back investment. Similarly, an increase in external costs of financing ϕ ought to induce firms to increase their precautionary cash holdings and to scale back their capital expenditures. This is exactly what our model predicts, as shown in Panel C and D. The effect of an increase in the carry cost λ ought to be to induce firms to spend their cash more readily, by disbursing it more frequently to shareholders or investing more aggressively. Interestingly, although cash holdings decrease with λ, as seen in Panel E, the net effect on investment is negative, as Panel F shows. A higher λ makes it more expensive for firms to maintain its buffer-stock cash holdings and indirectly raises the cost of investment.

Finally, one clear difference between Figure 6 and 7 is that, unlike the physical parameters, the parameters σ, ϕ, λ have rather limited effects on investment. This result implies that firms can

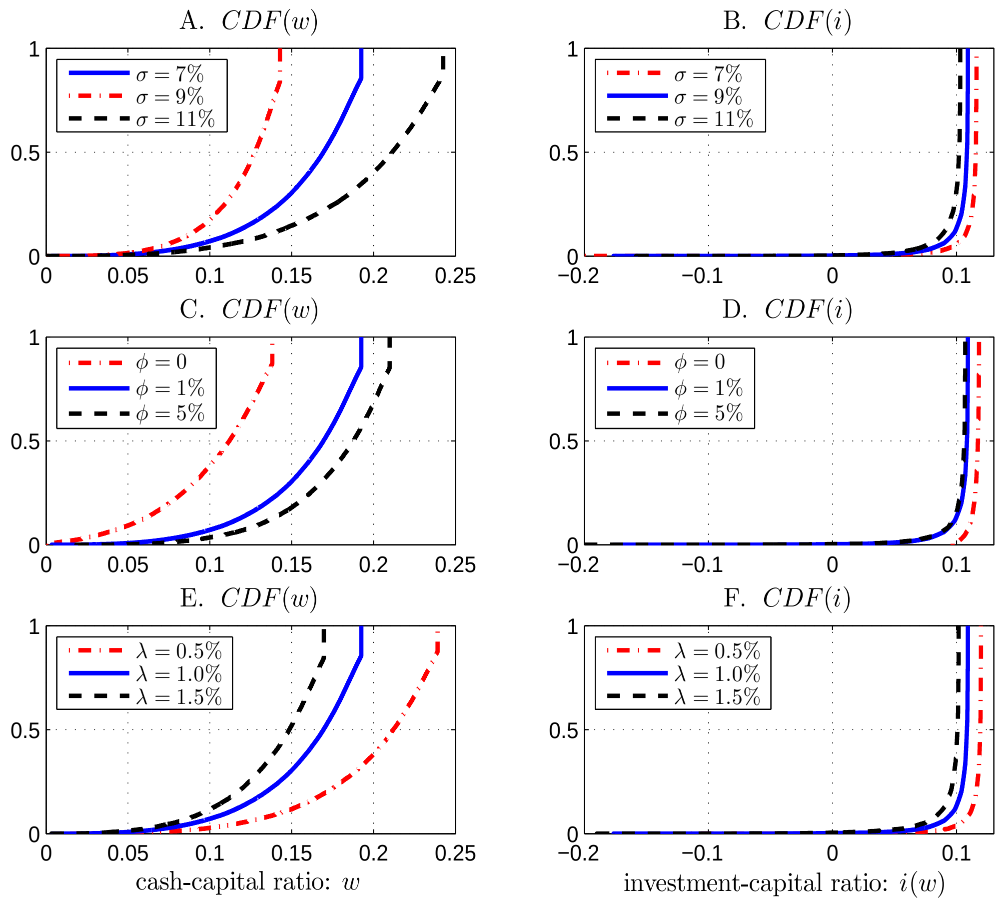

Figure 7: **Comparative statics II: σ, ϕ, and λ.** This figure plots the cumulative distribution function for the stationary distribution of cash-capital ratio (w) and investment-capital ratio ($i(w)$) for different values of the volatility of productivity shocks σ, fixed costs of external financing ϕ, and carry cost of cash λ.

effectively adjust their cash/payout/financing policies in response to changes in financing or cash management costs, limiting the impact on the real side (investment).

VI. Risk and Return

In this section, we investigate how the firm's investment, financing, and cash management policies affect the risk and return of the firm. In order to highlight the impact of financing constraints on the firm's risk and returns, we adopt the benchmark asset pricing model (CAPM), which measures the riskiness of an asset with its market beta. We use r_m and σ_m to denote the expected return and volatility of the market portfolio.

Without financial frictions (the MM world), the firm implements the first-best investment policy. Its expected return is constant and is given by the classical CAPM formula:

$$\mu^{FB} = r + \beta^{FB} \left(r_m - r \right), \tag{24}$$

where

$$\beta^{FB} = \frac{\rho\sigma}{\sigma_m} \frac{1}{q^{FB}}, \tag{25}$$

and ρ is the correlation between the firm's productivity shock dA and returns of the market portfolio.

We can derive an analogous *conditional* CAPM expression for the instantaneous expected return $\mu^r(w)$ of a *financially constrained firm* by applying Ito's lemma (see Duffie (2001)):

$$\mu^r(w) = r + \beta(w) \left(r_m - r \right), \tag{26}$$

where

$$\beta(w) = \frac{\rho\sigma}{\sigma_m} \frac{p'(w)}{p(w)} \tag{27}$$

is the conditional beta of the financially constrained firm.

Our analysis highlights how idiosyncratic risk affects the beta of a financially constrained firm. Idiosyncratic risk, as systematic risk, causes earnings fluctuations and induces underinvestment for firm facing external financing costs. Thus, through its effect on $p(w)$ and $p'(w)$, idiosyncratic risk affects beta.

Equation (27) implies that the beta for a financially constrained firm is monotonically decreasing with its cash-capital ratio w. The cash-capital ratio w has two effects on the conditional beta: first, an increase in w relaxes the firm's financing constraint and reduces underinvestment. As a result, the risk of holding the firm is lower. Second, the firm's asset risk is also reduced as a result of the firm holding a greater share of its assets in cash (whose beta is zero). Both channels imply that the conditional beta $\beta(w)$ and the required rate of return $\mu^r(w)$ decreases with w.

Interestingly, when w is sufficiently high, the beta for a firm facing external financing costs can be even lower than the beta for the neoclassical firm (facing no financing costs). We may illustrate this point by rewriting the conditional beta as follows:

$$\beta(w) = \frac{\rho\sigma}{\sigma_m} \frac{p'(w)}{(p(w) - w) + w} = \frac{\rho\sigma}{\sigma_m} \frac{p'(w)}{q_a(w) + w}, \tag{28}$$

where $q_a(w) = p(w) - w$ is the firm's *average* q (the ratio of the firm's *enterprise value* and its capital stock). Although $q_a(w) < q^{FB}$ and $p'(w) > 1$, the second term, w, in the denominator of $\beta(w)$ can be so large that $\beta(w) < \beta^{FB}$. Intuitively, as a financially constrained firm hoard cash to reduce external financing costs, the firm beta becomes a weighted average of the asset beta and the beta of cash (zero). With a large enough buffer stock of cash holdings relative to its assets, this firm can be even safer than neoclassical firms facing no financing costs and holding no cash.

Panel A of Figure 8 plots the firm's value-capital ratio $p(w)$ for three different levels of idiosyncratic volatility (5%, 15%, 30%). The other parameter values for this calculation are $r_m - r_f = 6\%$, $\sigma_m = 20\%$, and the systematic volatility is fixed at $\rho\sigma = 7.2\%$ (assuming $\rho = 0.8$ and $\sigma = 9\%$). As expected, it shows that firm value is higher and the payout boundary $\overline{w}$ is lower for lower levels of idiosyncratic volatility.

Panel B plots the *marginal value of cash* $p'(w)$ for the same three levels of idiosyncratic volatility. It shows, as expected, that $p'(w)$ is decreasing in w for each level of idiosyncratic volatility. The figure also reveals that for high values of w, the marginal value of cash $p'(w)$ is higher for higher levels of idiosyncratic volatility. But, more surprisingly, for low values of w the marginal value of cash is actually decreasing in idiosyncratic volatility. The reason is that when the firm is close to financial distress, a dollar is more valuable for a firm with lower idiosyncratic volatility, which is more likely to avoid raising external funds.

Panel C plots the *investment-capital ratio* for the three different levels of idiosyncratic volatility. We see again that for sufficiently high w, investment is decreasing in idiosyncratic volatility, whereas for low w, it is increasing. That is, when w is low, firms with low idiosyncratic volatility engage in more asset sales. Again, this latter result is driven by the fact that a marginal dollar has a higher

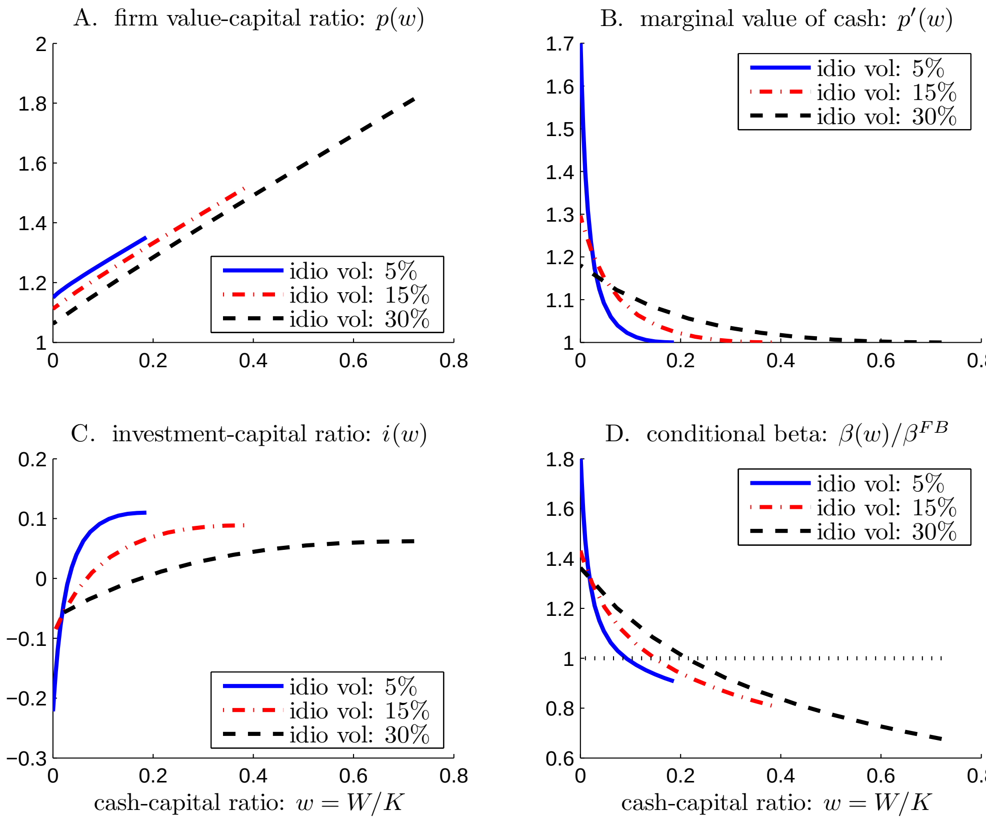

Figure 8: **Idiosyncratic volatility, firm value, investment, and beta.** In the refinancing case ($\phi = 1\%$), fixing all other parameters while using three different levels of idiosyncratic volatility $(5\%, 15\%, 30\%)$, this figure plots the firm value-capital ratio, marginal value of cash, investment-capital ratio, and the ratio of the conditional beta of a constrained firm to that of an unconstrained firm (first best). The right end of each line corresponds to the respective payout boundary.

value for a firm with lower idiosyncratic volatility. Therefore, such a firm will sell more assets to replenish its cash holdings.

Panel D plots *conditional betas* normalized by the first-best beta: $\beta(w)/\beta_{FB}$. For the same firm, $\beta(w)$ is decreasing in the cash-capital ratio w. At low levels of w, the firm's normalized beta $\beta(w)/\beta_{FB}$ can approach a value as high as 1.8 for idiosyncratic volatility of 5%. On the other hand, $\beta(w)$ is actually lower than β^{FB} for high w. For example, the conditional beta can be as low as 60% of the first-best beta in the case of 30% idiosyncratic volatility. As we have explained above, this is due to the fact that a financially constrained firm endogenously hoards significant amounts of cash, a perfectly safe asset, so that the mix of a constrained firm's assets may actually be safer than the asset mix of an unconstrained firm, which does not hoard any cash.

The rankings of beta across the firms with different idiosyncratic volatility depends on w. For large cash-capital ratio w, the beta is increasing in the idiosyncratic volatility. However, when the level of w is low, firms with low idiosyncratic volatility actually have higher beta. The rankings of beta are driven by the ratio $p'(w)/p(w)$, which can be inferred from the top two panels.

These observations from Panel D have important implications about cross sectional studies of betas and cash holdings for financially constrained firms. First, after controlling for technology parameters and financing costs, the model predicts an inverse relation between returns and corporate cash holdings, which has been documented by Dittmar and Mahrt-Smith (2007) among others. Our analysis points out that this *negative relation* may not just be due to agency problems, as they emphasize, but may also be driven by the changing asset risk composition of the firm.

Second, for a cross section of firms with heterogeneous production technologies and external financing costs, it is crucial to take into account the endogeneity of cash holdings when we compare firm betas. As we have seen, a constrained firm's beta can be either higher or lower than the beta of an unconstrained firm. Similarly, a firm with high external financing costs is more likely to hold a lot of cash, but its conditional beta (and expected return) may still be higher than for a firm with low financing costs and low cash holdings. Thus, a *positive relation* between returns and corporate cash holdings in the cross section of heterogenous firms may still be consistent with our model (see Palazzo (2009) for a related model and supporting empirical evidence).

VII. Dynamic Hedging

In addition to cash inventory management, the firm can also reduce its cash flow risk by investing in financial assets (an aggregate market index, options, or futures contracts) which are correlated with its own business risk. Consider, for example, the firm's hedging policy using market index futures. Let F denote the futures price. Under the risk-adjusted probability, the futures price evolves according to:

$$dF_t = \sigma_m F_t dB_t, \tag{29}$$

where σ_m is the volatility of the market portfolio, and B_t is a standard Brownian motion that is partially correlated with Z_t (with correlation coefficient ρ).

Let ψ_t denote the fraction of total cash W_t that the firm invests in the futures contract. Futures contracts often require that the investor hold cash in a margin account, which is costly. Let κ_t denote the fraction of the firm's total cash W_t held in the margin account ($0 \le \kappa_t \le 1$). In addition to the carry cost as cash in the standard interest-bearing account, cash held in this margin account also incurs the additional flow cost ϵ per unit of cash. We assume that the firm's futures position (in absolute value) cannot exceed a constant multiple π of the amount of cash $\kappa_t W_t$ in the margin account.[21] That is, we require

$$|\psi_t W_t| \le \pi \kappa_t W_t. \tag{30}$$

As the firm can costlessly reallocate cash between the margin account and its regular interest-bearing account at any time, the firm will optimally hold the minimum amount of cash necessary in the margin account. That is, provided that $\epsilon > 0$, optimality implies that the inequality (30) holds as an equality. When the firm takes a hedging position in a futures index, its cash then evolves as follows:

$$dW_t = K_t\left(\mu dA_t + \sigma dZ_t\right) - \left(I_t + G_t\right)dt + dH_t - dU_t + (r-\lambda)W_t dt - \epsilon\kappa_t W_t dt + \psi_t \sigma_m W_t dB_t. \tag{31}$$

Before analyzing optimal firm hedging constrained by costly margin requirements, we first investigate the case where there are no margin requirements for hedging.

A. Optimal Hedging with No Frictions

With no margin requirement ($\pi = \infty$), the firm carries all its cash in the regular interest-bearing account and is not constrained in the size of the futures positions ψ. Firm value $P(K, W)$ then

[21] For simplicity, we abstract from any variation of margin requirement, so that π is constant.

solves the following HJB equation:

$$\begin{aligned} rP(K,W) = \max_{I,\psi} \ & (I-\delta K)\, P_K(K,W) + ((r-\lambda)W + \mu K - I - G(I,K))\, P_W(K,W) \\ & + \frac{1}{2}\left(\sigma^2 K^2 + \psi^2\sigma_m^2 W^2 + 2\rho\sigma_m\sigma\psi W K\right) P_{WW}(K,W) \end{aligned} \tag{32}$$

The only difference between (32) and the HJB equation (9) with no hedging is the coefficient of the volatility term (the last term on the second line), which is now affected by hedging. Since firm value $P(K,W)$ is concave in W, so that $P_{WW} < 0$, the optimal hedging position ψ is determined simply by minimizing that coefficient with respect to ψ. The FOC for ψ is:

$$\left(\psi\sigma_m^2 W^2 + \rho\sigma_m\sigma W K\right) P_{WW} = 0.$$

Solving for ψ, we obtain the firm's optimal hedging demand:

$$\psi^*(w) = -\frac{\rho\sigma}{w\sigma_m}. \tag{33}$$

Thus, controlling for size (capital K), the firm hedges more when its cash-capital ratio w is low. Intuitively, the benefit of hedging is greater when the marginal value of cash $p'(w)$ is high. Substituting $\psi^*(w)$ into the HJB equation (32) and exploiting homogeneity, we obtain the following ODE for the firm's value-capital ratio under hedging:

$$rp(w) = (i(w)-\delta)\left(p\,(w) - wp'\,(w)\right) + ((r-\lambda)w + \mu - i(w) - g(i(w)))\, p'\,(w) + \frac{\sigma^2\left(1-\rho^2\right)}{2} p''\,(w)\,. \tag{34}$$

Note that the ODE above is the same as (13) in the case without hedging except for the variance reduction from σ^2 to $\sigma^2(1-\rho^2)$.

In sum, with frictionless hedging (no margin requirements and $\epsilon = 0$), the firm completely eliminates its systematic risk exposure via hedging. The firm thus behaves exactly in the same way as the firm in our baseline model of Section II with only idiosyncratic volatility $\sigma\sqrt{1-\rho^2}$.

B. Optimal Hedging with Margin Requirements

Next, we consider the more realistic setting with a margin requirement given by (30). The firm then faces both a cost of hedging and a constraint on the size of its hedging position. As a result, the firm's HJB equation now takes the following form:

$$rP(K,W) = \max_{I,\psi,\kappa} (I - \delta K) P_K(K,W) + ((r-\lambda)W + \mu K - I - G(I,K) - \epsilon\kappa W) P_W(K,W) + \frac{1}{2}\left(\sigma^2 K^2 + \psi^2 \sigma_m^2 W^2 + 2\rho\sigma_m\sigma\psi W K\right) P_{WW}(K,W) \quad (35)$$

subject to:

$$\kappa = \min\left\{\frac{|\psi|}{\pi}, 1\right\}. \quad (36)$$

Equation (36) indicates that there are two candidate solutions for κ (the fraction of cash in the margin account): one interior and one corner. If the firm has sufficient cash, so that its hedging choice ψ is not constrained by its cash holding, the firm sets $\kappa = |\psi|/\pi$. This choice of κ minimizes the cost of the hedging position subject to meeting the margin requirement. Otherwise, when the firm is short of cash, it sets $\kappa = 1$, thus putting all its cash in the margin account to take the maximum feasible hedging position: $|\psi| = \pi$.

The direction of hedging (long ($\psi > 0$) or short ($\psi < 0$)) is determined by the correlation between the firm's business risk and futures return. With $\rho > 0$, the firm will only consider taking a short position in futures as we have shown. If $\rho < 0$, the firm will only consider taking a long position. Without loss of generality, we focus on the case where $\rho > 0$, so that $\psi < 0$.

First, consider the cash region with an interior solution for ψ (where the fraction of cash allocated to the margin account is given by $\kappa = -\psi/\pi < 1$). The FOC with respect to ψ is:

$$\frac{\epsilon}{\pi} W P_W + \left(\sigma_m^2 \psi W^2 + \rho\sigma_m\sigma W K\right) P_{WW} = 0.$$

Using homogeneity, we may simplify the above equation and obtain:

$$\psi^*(w) = \frac{1}{w}\left(\frac{-\rho\sigma}{\sigma_s} - \frac{\epsilon}{\pi}\frac{p'(w)}{p''(w)}\frac{1}{\sigma_s^2}\right). \tag{37}$$

Consider next the low cash region. The benefit of hedging is high in this region ($p'(w)$ is high when w is small). The constraint $\kappa \leq 1$ is then binding, hence $\psi^*(w) = -\pi$ for $w \leq w_-$, where the endogenous cutoff point w_- is the unique value satisfying $\psi^*(w_-) = -\pi$ in (37).

Finally, when w is sufficiently high, the firm chooses not to hedge, as the net benefit of hedging approaches zero while the cost of hedging remains bounded away from zero. More precisely, we have $\psi^*(w) = 0$ for $w \geq w_+$, where the endogenous cutoff point w_+ is the unique solution of $\psi^*(w_+) = 0$ using (37).

In summary, there are three endogenously determined regions for optimal hedging. For sufficiently low cash ($w \leq w_-$), the firm engages in maximum feasible hedging ($\psi(w) = -\pi$). All the firm's cash is in the margin account. In the (second) interior region $w_- \leq w \leq w_+$, the firm chooses its hedge ratio $\psi(w)$ according to equation (37) and puts up just enough cash in the margin account to meet the requirements. For high cash holdings ($w \geq w_+$), the firm does not engage in any hedging to avoid the hedging costs.

We now provide quantitative analysis of the impact of hedging on the firm's decision rules and firm value. We choose the following parameter values: $\rho = 0.8$, $\sigma_m = 20\%$ (the same as in Section VI); $\pi = 5$, corresponding to 20% margin requirement; $\epsilon = 0.5\%$; the remaining parameters are those for the baseline case in Section V.

In Figure 9, several striking observations emerge from the comparisons of the *frictionless hedging*, the *hedging with costly margin requirements*, and the *no hedging* solutions.

First, Panel A makes apparent the extent to which hedging may be constrained by the margin requirements. On the one hand, when $w > w_+ = 0.11$, the firm chooses not to hedge at all because the benefits of hedging are smaller than the costs due to margin requirements. On the other hand, it hits the maximum hedge ratio for $w < w_- = 0.07$. Thus, just when hedging is most valuable, the firm will be significantly constrained in its hedging capacity. As a result, the firm effectively

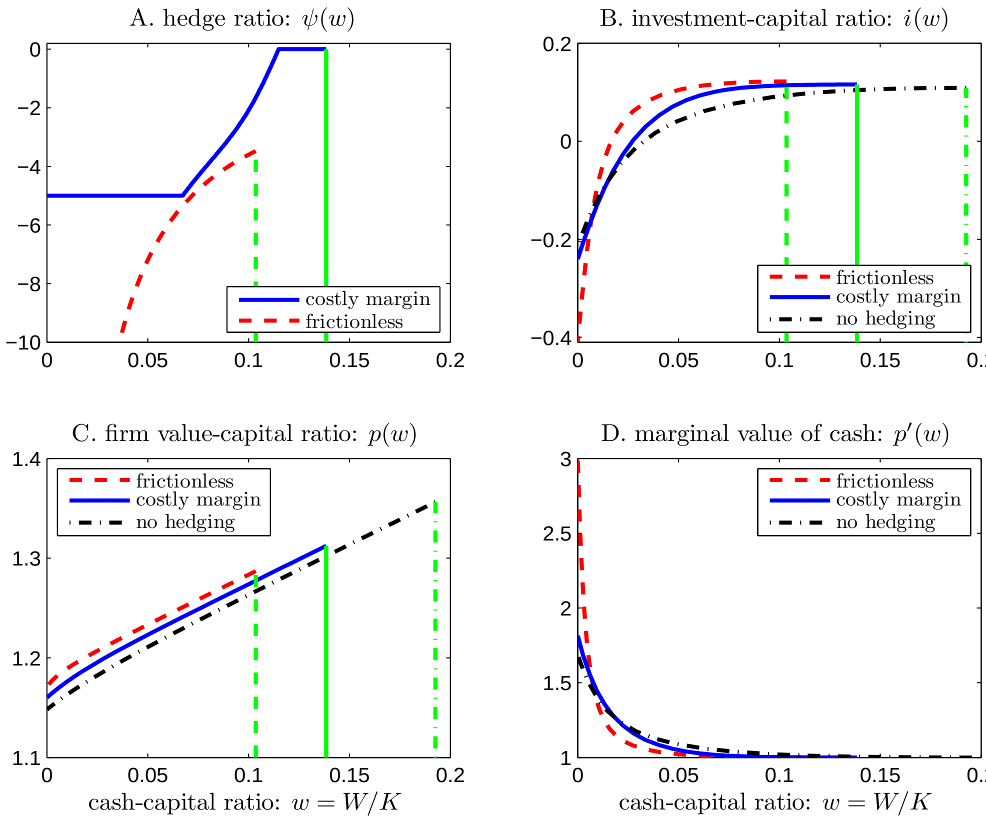

Figure 9: **Optimal hedging.** This figure plots the optimal hedging and investment policies, the firm value-capital ratio, and the marginal value of cash for Case II with hedging (with or without margin requirements). In Panel A, the hedge ratio for the frictionless case is cut off at -10 for display. The right end of each line corresponds to the respective payout boundary.

faces higher uncertainty under costly hedging than under frictionless hedging. It follows that the firm chooses to postpone payouts to shareholders (the endogenous upper boundary $\overline{w}$ shifts from 0.10 to 0.14). The firm also optimally scales back its hedging position in the middle region due to the costs of hedging.

Second, Panel B reveals the surprising result that for low cash-capital ratios, the firm may *underinvest* even more when it is able to optimally hedge (whether with or without costly margin requirements) than when it cannot hedge at all. This is surprising, as one would expect the firm's underinvestment problem to be mitigated by hedging. After all, hedging reduces the firm's earnings volatility and thus should reduce the need for precautionary cash balances. This rough intuition is partially correct as, indeed, the firm does invest more for sufficiently high values of w, when it engages in hedging.

But why should the firm invest less or disinvest more for low values of w? The reason can be found in Panels C and D. Panel C plots $p(w)$ under the three settings and confirms the intuition that hedging increases firm value. As expected, $p(w)$ is highest under frictionless hedging and lowest without hedging. However, remarkably, not only is $p(w)$ higher with hedging, but the marginal value of cash $p'(w)$ is also higher, when w is low. Panel D plots the marginal values of cash under the three solutions. Observe that the marginal value of cash is actually higher for low values of w, when the firm engages in hedging. With a higher marginal value of cash, it is then not surprising that the firm sells its assets more aggressively and hedges its operation risk in order to lower the likelihood of using costly external financing.

How much value does hedging add to the firm? We answer this question by computing the net present value (NPV) of optimal hedging to the firm for the case with costly margin requirements. The NPV of hedging is defined as follows. First, we compute the *cost of external financing* as the difference in Tobin's q under the first-best case and q under Case II without hedging. Second, we compute the *loss in adjusted present value* (APV), which is the difference in the Tobin's q under the first-best case and q under Case II with costly margin. Then, the difference between the costs of external financing and the loss in APV is simply the value created through hedging. On average, when measured relative to Tobin's q under hedging with a costly margin, the costs of external financing is about 6%, the loss in APV is about 5%, so that the NPV of costly hedging is of the order of 1%, a significant creation of value to say the least for a purely financial operation.

VIII. Credit Line

Our baseline model of Section II can also be extended to allow the firm to draw down a credit line. This is an important extension to consider, as many firms in practice are able to secure such lines, and for these firms, access to a credit line is an important alternative source of *liquidity* than cash.

We model the credit line as a source of funding the firm can draw on at any time it chooses up to a limit. We set the credit limit to a maximum fraction of the firm's capital stock, so that the firm can borrow up to cK, where $c > 0$ is a constant. The logic behind this assumption is that the firm must be able to post collateral to secure a credit line and the highest quality collateral does not exceed the fraction c of the firm's capital stock. We may thus interpret cK to be the firm's short-term debt capacity. We also assume that the firm pays a constant spread α over the risk-free rate on the amount of credit it uses. That is, the firm pays interest on its credit at the rate $r + \alpha$. Sufi (2009) shows that a firm on average pays 150 basis points over LIBOR on its credit lines. This essentially completes the description of a credit line in our model. We leave other common clauses of credit lines-such as commitment fees and covenants-as well as the endogenous determination of the limit cK to future research.

Since the firm pays a spread α over the risk-free rate to access credit, it will optimally avoid using its credit line or other costly external financing before exhausting its internal funds (cash) to finance investment. The firm does not pay fixed costs in accessing the credit line, so it also prefers to first draw on the line before tapping equity markets as long as the interest rate spread α is not too high.[22] Our model thus generates a pecking order among internal funds, credit lines and external equity financing.

As in the baseline model, in the cash region, the firm value-capital ratio $p(w)$ satisfies the ODE in (13), and has the same boundary conditions for payout (16-17). When credit is the marginal source of financing (credit region), $p(w)$ solves the following ODE:

$$rp(w) = (i(w) - \delta) \left(p(w) - wp'(w)\right) + ((r + \alpha)w + \mu - i(w) - g(i(w)))\, p'(w) + \frac{\sigma^2}{2} p''(w) , \quad w < 0 \tag{38}$$

When the firm exhausts its credit line before issuing equity, the boundary conditions for the timing and the amount of equity issuance are similar to the ones given in Section IV. That is, we have $p(-c) = p(m) - \phi - (1 + \gamma)(m + c)$, and $p'(m) = 1 + \gamma$. Finally, $p(w)$ is continuous and smooth everywhere, including at $w = 0$, which gives two additional boundary conditions.

Figure 10 plots the firm's value-capital ratio $p(w)$, the marginal value of liquidity $p'(w)$, the

[22]When α is high and equity financing costs (ϕ, γ) are low, the firm may not exhaust its credit line before accessing external equity markets. For our parameter values, we find that the pecking order results apply between the credit line and external equity.

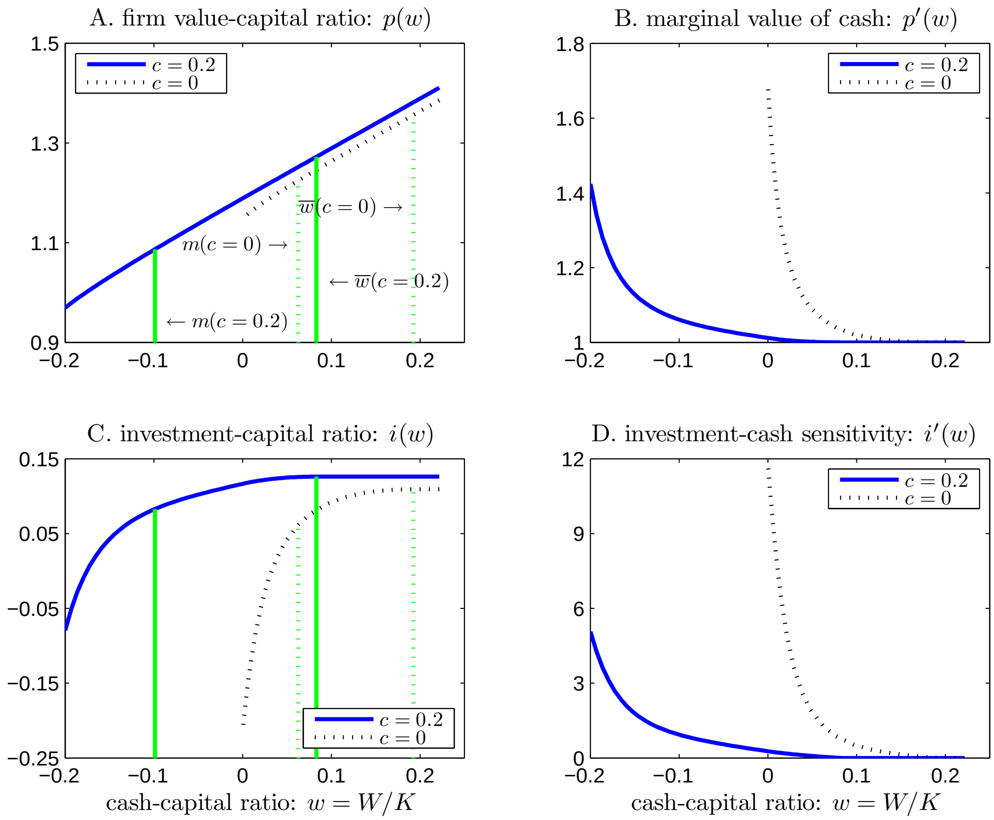

Figure 10: **Credit line.** This figure plots the model solution with credit line and external equity. Each panel plots for two scenarios: one without credit line ($c = 0$) and the other with credit line ($c = 20\%$). The spread on the credit line is $\alpha = 1.5\%$ over the risk-free rate r.

investment-capital ratio $i(w)$, and the investment-cash sensitivity $i'(w)$, when the firm has access to a credit line. As can be seen from the figure, having access to a credit line increases $p(w)$. This is to be expected, as access to a credit line provides a cheaper source of external financing than equity under our chosen parameter value for the spread on the credit line: $\alpha = 1.5\%$. Second, observe that with the credit line option the firm hoards significantly less cash, and the payout boundary $\overline{w}$ drops from 0.19 to 0.08 when the credit line increases from $c = 0$ to $c = 20\%$ of the firm's capital stock. Third, without access to a credit line ($c = 0$), the firm raises lumpy amounts of equity mK (with $m = 0.06$ for $\phi = 1\%$) when it runs out of cash. In contrast, when $c = 20\%$, the firm raises $0.1K$ in a new equity offering when it has exhausted its credit line, so as to pay off most of the debt it has accumulated on its credit line. But, note that for our baseline parameter choices, the firm still remains in debt after the equity issuance, as $m = -0.10$. Fourth, the credit line substantially lowers the marginal value of liquidity. Without the credit line, the marginal value of cash at $w = 0$ is $p'(0) = 1.69$, while with the credit line ($c = 20\%$), the marginal value of cash at $w = 0$ is $p'(0) = 1.01$, and the marginal value of cash at the point when the firm raises external equity is $p'(-c) = 1.42$.

It follows that a credit line substantially mitigates the firm's underinvestment problem as can be seen in Panel C in Figure 10. Without a credit line ($c = 0$), the firm engages in significant asset sales ($i = -21.4\%$) when it is about to run out of cash. With a credit line, however ($c = 20\%$), the firm's investment-capital ratio is $i(0) = 11.7\%$ when it runs out of cash ($w = 0$). Even when the firm has exhausted its credit line (at $w = -20\%$), it engages in much less costly asset sales ($i(-c) = -7.9\%$). Finally, observe that the investment-cash sensitivity is substantially lower when the firm has access to a credit line. For example, when the firm runs out of cash, the investment-cash sensitivity is only $i'(0) = 0.27$, much smaller than when the firm has no credit line and has to issue external equity to finance investment ($i'(0) = 11.8$).

Next, we turn to the effect of liquidity (cash and credit) on average q, marginal q, and investment.

The left panel of Figure 11 plots the firm's marginal q and average q for two otherwise identical firms: one with a credit line ($c = 20\%$), and the other without a credit line ($c = 0$). First we see that average q increases with w in both credit and cash regions, because $q_a'(w) = p'(w) - 1 \geq 0$. The inequality follows from the result that the marginal value of liquidity $p'(w) \geq 1$.

Second, recall that marginal q is related to average q as follows in both regions:

$$q_m(w) = q_a(w) - (p'(w) - 1)w.$$

When the firm is in the cash region, marginal q lies below average q, because $p'(w) \geq 1$ and $w > 0$. The intuition is that a unit increase in capital K lowers the firm's cash-capital ratio $w = W/K$, which causes the firm to be more financially constrained, thus making marginal q lower than average q. In contrast, when the firm is in the credit region ($w < 0$), increasing K raises the firm's debt

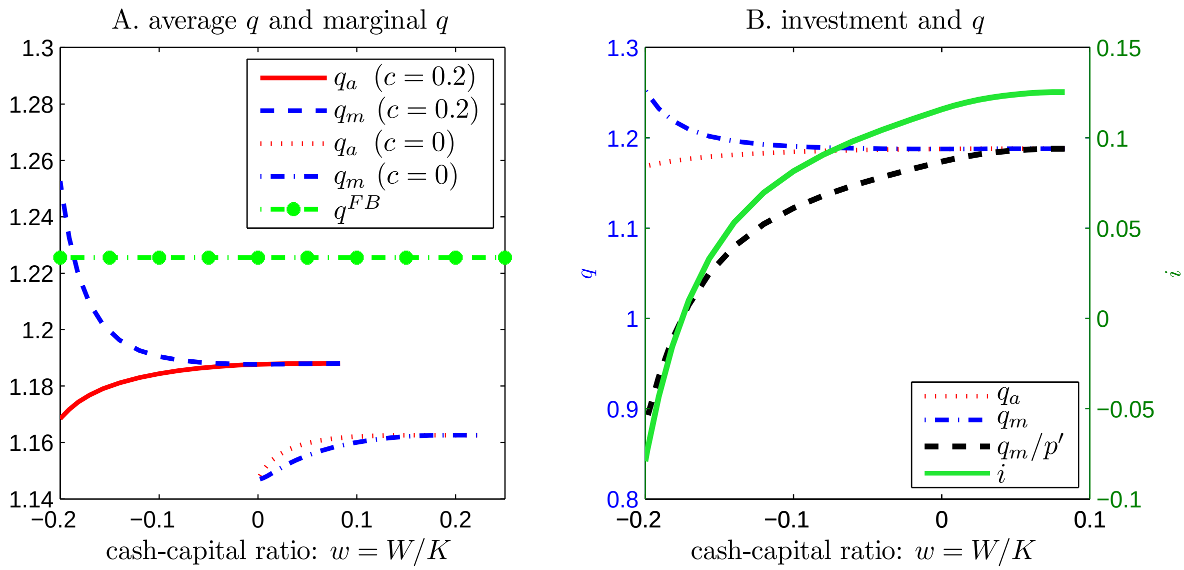

Figure 11: **Investment and q with credit line.** The left panel plots the average q (q_a) and marginal q (q_m) from the case with credit line ($c = 0.2$) and without credit line ($c = 0$) up to the respective payout boundaries. The right panel plots the average q, marginal q, the ratio of marginal q to marginal value of liquidity (q_m/p') on the left axis, and investment-capital ratio (i) on the right axis. These results are for the case with credit line ($c = 0.2$).

capacity (credit line limit cK) and lowers its leverage, which relaxes the firm's borrowing constraint. This effect causes marginal q to be larger than average q for $w < 0$.

While both average q and investment $i(w)$ are increasing in w in both credit and cash regions, marginal q is not monotonic in w. This can be seen from the following:

$$q_m'(w) = -p''(w)w.$$

Because w can be either signed, marginal q decreases in w when $w > 0$ and increases in w when $w < 0$. Moreover, while average q is always below the first-best q, marginal q may exceed the first-best marginal q when the firm is in the credit region (due to the debt capacity channel), as seen in Figure 11. We also observe that the quantitative differences between average and marginal q are much larger in the credit region than in the cash region.

It is sometimes argued that when there are no fixed costs of investment marginal q is a more

Table II: CONDITIONAL MOMENTS FROM THE STATIONARY DISTRIBUTION OF THE CREDIT LINE MODEL

This table reports the population moments for cash-capital ratio (w), investment-capital ratio ($i(w)$), marginal value of cash ($p'(w)$), average q ($q_a(w)$), and marginal q ($q_m(w)$) from the stationary distribution in the case with credit line.

	w	$i(w)$	$p'(w)$	$q_a(w)$	$q_m(w)$
			A. credit region		
mean	-0.040	0.104	1.030	1.188	1.190
median	-0.030	0.108	1.023	1.188	1.189
std	0.034	0.015	0.023	0.001	0.002
skewness	-1.214	-3.071	4.186	-3.387	9.011
kurtosis	4.228	19.200	34.462	20.552	125.634
			B. cash region		
mean	0.055	0.124	1.002	1.189	1.189
median	0.060	0.125	1.001	1.189	1.189
std	0.024	0.002	0.003	0.000	0.000
skewness	-0.569	-1.602	1.636	-2.146	-0.860
kurtosis	2.219	4.710	4.841	6.950	2.388

accurate measure than average q for the firm's investment opportunities.[23] This is indeed true in the MM world but it is not generally valid in a world where firms face financial constraints. The right panel of Figure 11 shows that although the investment-capital ratio $i(w)$ increases with w in the credit region, marginal q actually decreases with w. As a result, marginal q and investment move in opposite directions in the credit region. To understand this seemingly counterintuitive result, we must look at the investment Euler equation for a firm facing financial constraints. It is clear from this equation that investment is driven by the ratio of marginal q to the marginal value of liquidity $p'(w)$. Now in the credit region, both the marginal value of liquidity $p'(w)$ and marginal q are high when the firm is close to its credit limit. Indeed, the marginal value of liquidity $p'(w)$ increases at a higher rate than marginal q when the firm uses up more of its credit line (i.e. when we move to the left in the credit region), and as a result, the investment-capital ratio falls when the firm uses more credit.

[23]Caballero and Leahy (1996) show that average q can be a better proxy for investment opportunities in the presence of fixed costs of investment.

Finally, we turn to the analysis of the stationary distribution of firms with access to credit line. To understand the different behavior in the cash and the credit regions, we report the first four moments of the distribution plus the medians of the variables of interests (w, $i(w)$, $p'(w)$, $q_a(w)$, and $q_m(w)$) for both the credit region and cash region in Table II. The most significant observation is that the availability of credit makes the firm's stationary distribution for these variables much less skewed and fat-tailed in the cash region. Because liquidity is more abundant with a credit line, the firm's marginal value of cash is effectively unity throughout the cash region. However, the skewness and fat-tails of the distribution now appear in the credit region (note, for example, the high kurtosis (126) for marginal q in the credit region). Although the firm has a credit line of up to 20% of its capital stock, it only uses about 4% of its line on average. The reason is that the firm does not spend much time around the credit line limit. The risk of facing a large fixed cost of equity induces the firm to immediately move away from its credit limit.[24]

The cash-capital ratio w, $i(w)$, $q_a(w)$, and $q_m(w)$ are all skewed to the left in the cash region, as in our baseline model without a credit line. The intuition is similar to the one provided in the baseline model. Moreover, because of the firm's optimal buffer-stock cash holding, there is effectively no variation in the cash region for the firm's investment and value. Note also that the mean and median of marginal q and average q are all equal to 1.189, up to the third decimal point. Even for the investment-capital ratio $i(w)$, the difference between its median and mean values only appear at the third decimal point.

Unlike in the cash region, not only is the marginal value of credit $p'(w)$ skewed to the left, but so is marginal q in the credit region. The left skewness of marginal q and $p'(w)$ are both driven by the fact that every so often the firm hits the credit limit and incurs large financing costs. In other words, there is much more variation in the credit region than in the cash region for marginal q and the marginal value of liquidity $p'(w)$. As marginal q and the marginal value of liquidity move in the same direction in the credit region, there is, however, much less variation in $i(w)$, which is monotonically related to the ratio $q_m(w)/p'(w)$.

[24]DeAngelo, DeAngelo and Whited (2009) make a similar observation.

IX. Conclusion

We introduce external financing costs, an important friction emphasized in modern corporate finance literature, into the neoclassic q theory of investment. Using a tractable and operational dynamic economic framework, we show how the firm's optimal investment, financing, and risk management policies are interconnected in the presence of external financing costs. In our model, corporate risk management involves internal liquidity management, financial hedging, investment/asset sales, and payout. Several new insights emerge from our analysis. For example, we find that the relation between marginal q and investment differs depending on whether cash or credit is the marginal source of financing. We also demonstrate the distinct and complementary roles that cash management and derivatives play in risk management.

Our model can be extended to have time-varying investment and financing opportunities, as well as endogenous leverage decisions. Allowing for stochastic financing opportunities may generate rational "market-timing" of financing. As our analysis only looks at risk management in a reduced-form agency model, it would clearly be desirable explore a model where decision-making by an incentivized self-interested manager is explicitly modeled.[25] Our dynamic tradeoff model does not explicitly capture the effects of taxes on risk management (see Smith and Stulz (1985) and Graham and Smith (1999) for early static theory and empirical evidence, respectively). Neither do we model the impact of strategic considerations, such as building a war-chest to improve the firm's competitive position in product markets, on firms' cash-inventory and risk management decisions (see Haushalter, Klasa, and Maxwell (2007) and Harford (1999) for empirical evidence). We leave these extensions to future research.

[25] Pinkowitz, Stulz and Williamson (2006) and Dittmar and Mahrt-Smith (2007) empirically explores the relation between the firms' excess cash holdings and corporate governance. See Dittmar (2008) for a survey of this literature.

Appendix

Boundary conditions. We begin by showing that $P_W(K, W) \geq 1$. The intuition is as follows. The firm always can distribute cash to investors. Given $P(K, W)$, paying investors $\zeta > 0$ in cash changes firm value from $P(K, W)$ to $P(K, W - \zeta)$. Therefore, if the firm chooses not to distribute cash to investors, firm value $P(K, W)$ must satisfy

$$P(K, W) \geq P(K, W - \zeta) + \zeta,$$

where the inequality describes the implication of the optimality condition. With differentiability, we have $P_W(K, W) \geq 1$ in the accumulation region. In other words, the marginal benefit of retaining cash within the firm must be at least unity due to costly external financing. Let $\overline{W}(K)$ denote the threshold level for cash holding, where $\overline{W}(K)$ solves

$$P_W\left(K, \overline{W}(K)\right) = 1. \tag{39}$$

The above argument implies the following payout policy:

$$dU_t = \max\{W_t - \overline{W}(K_t), 0\},$$

where $\overline{W}(K)$ is the endogenously determined payout boundary. Note that paying cash to investors reduces cash holding W and involves a linear cost. The following standard condition, known as *super contact* condition, characterizes the endogenous upper cash payout boundary (see e.g. Dumas, 1991 or Dixit, 1993):

$$P_{WW}(K, \overline{W}(K)) = 0. \tag{40}$$

When the firm's cash balance is sufficiently low ($W \leq \underline{W}$), under-investment becomes too costly. The firm may thus rationally increase its internal funds to the amount $\underline{W}$ by raising total amount of external funds $(1+\gamma)\left(\underline{W}-W\right)$. Optimality implies that

$$P(K,W)=P(K,\underline{W})-(1+\gamma)\left(\underline{W}-W\right),\quad W\leq\underline{W}. \tag{41}$$

Taking the limit by letting $W\rightarrow\underline{W}$ in (41), we have

$$P_W(K,\underline{W}(K))=1+\gamma. \tag{42}$$

Numerical procedure. We use the following procedure to solve the free boundary problem specified by ODE (13) and the boundary conditions associated with the different cases. First, we postulate the value of the free (upper) boundary $\overline{w}$, and solve the corresponding initial value problem using the Runge-Kutta method. For each value of $\overline{w}$ we can compute the value of $p(w)$ over the interval $[0,\overline{w}]$. We can then search for the $\overline{w}$ that will satisfy the boundary condition for p at $w=0$. In the cases with additional free boundaries, including Case II and the model of hedging with margin requirements, we search for $\overline{w}$ jointly with the other free boundaries by imposing additional conditions at the free boundaries.

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