

Market Liquidity and Liquid Wealth

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Abstract

Are securities markets more liquid when the economy is more liquid? If so, why? One possibility is that market depth depends on credit constrained intermediaries. This paper offers another explanation, which does not involve frictions or market segmentation. Measuring market illiquidity by the slope of the representative agent's demand curve for a risky asset, I show that this slope is steeper when money-like investments (or liquid wealth) represent less of an economy's assets. That is because an exchange of risky shares for money in such a state induces greater intertemporal substitution than it does when there are more liquid balances. Agents are not indifferent to this substitution, and so prices respond more to trade. Thus market illiquidity fluctuates naturally with the level of real liquidity. This observation has important implication for understanding the causes of liquidity risk and market fragility.

KEYWORDS: CREDIT CONDITIONS, LIQUIDITY RISK, SAVINGS, ASSET PRICING.

JEL CLASSIFICATIONS: D91, E21, E44, E52, G12

1 Introduction

It is not obvious whether or why the ease of transacting in financial markets should be affected by the amount of liquid wealth in an economy. In times of economic or political turbulence, however, understanding the connection between market liquidity (the first notion) and balance-sheet liquidity (the second) may be of crucial policy importance.

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In such situations, preserving the orderly functioning of securities markets is tantamount to preventing extreme market illiquidity. That is, the policy objective facing authorities is not (presumably) to artificially prop up markets in the face of bad news, but rather to prevent them from “seizing up” or becoming so illiquid that even small trades require steep price concessions. This type of illiquidity may inhibit risk sharing and risk control by making trade prohibitively costly. It also may prevent markets from correctly transmitting price signals as trades take place at wide spreads and possibly at off-equilibrium values. Distorted prices, in turn, raise the specter of systemic real effects if informationless trades trigger solvency constraints at banks and other intermediaries, leading to broader asset disposals. Such “fire sales” themselves would not be problematic if the remaining unconstrained agents could absorb them. Again, it is not the level of the market, but its depth, that needs protecting. Dysfunctional markets may mean inefficient prices.

To combat market illiquidity, central banks will typically signal that they stand ready to “provide liquidity” to the financial system. Here the word liquidity is being used in a very different sense. The authorities do not themselves stand ready to make two-way prices in risky securities. Instead, the liquidity they provide is cash. More specifically, the Federal Reserve can expand the real monetary base by lending reserves directly or by open market operations (or by signalling that it is prepared to do both). These actions raise the percentage of private sector assets that are held in liquid form. Implicitly, then, central banks attempt to use one type of liquidity to affect the other. The question is: why should this work? What does the stock market’s willingness to accommodate trade have to do with the money supply?

This paper seeks to understand how market liquidity might be related to the economy’s liquidity. For the two do seem to be linked. Not only does the hypothesized crisis-period interaction seem grounded in fact,¹ but there is evidence that the liquidities are connected in normal times as well.²

A natural and widely-held view of the mechanics of this “liquidity substitution” is that, in varying the money supply, central banks are tightening or loosening the financ-

¹Chordia, Sarkar, and Subrahmanyam (2005) report that bid/ask spreads in stock and bond markets were negatively correlated with measures of monetary easing during three high-stress periods from 1994 to 1998.

²In monthly data from 1965 through 2001, Fujimoto (2004) finds that several measure of aggregate illiquidity are significantly lower during expansionary monetary regimes than during contractionary ones. Vector autoregressions also indicate a significant response of illiquidity to monetary or Fed Funds rate innovations, at least in the first half of the sample.

ing conditions which enable intermediaries to make markets for risky securities. This intuition presupposes the existence of a credit channel through which nominal quantities affect real financing conditions. It also implicitly relies on some sort of “inventory cost” model of price setting, whereby bid and ask prices are determined by a financially constrained sector whose cost of capital (a shadow cost when their constraint binds) differs from that of the economy as a whole. Finally, to the extent that intervention is viewed as beneficial, it must be that these constraints themselves are inefficient, perhaps resulting from agency problems or asymmetric information.

The view of financing constraints leading to imperfect intermediation has been thoroughly developed by Allen and Gale.³ Financially constrained intermediaries are also the crucial ingredient in the limits-to-arbitrage literature stemming from Shleifer and Vishny (1997). Gromb and Vayanos (2002) formally model the liquidity provision decision of arbitrageurs subject to a positive wealth financing constraint. Brunnermeier and Pedersen (2005) study the effect of a value-at-risk type constraint (whose tightness they call funding illiquidity) on price concessions demanded by risk-neutral market makers (a measure of market illiquidity) in a two-period game. The approach of this line of research is similar in spirit to classical microstructure models of liquidity provision in that the key features of the institutional setting (determining who is allowed to be an intermediary and how they may finance themselves) are taken as exogenous.

Here I present an alternative model of the effect of liquid balances on market liquidity, in which intermediaries play no role. The analysis is based upon the notion of market illiquidity introduced in Pagano (1989) which quantifies the aggregate price response to a perturbation in the supply of an asset. This quantity, which is equivalent to the (inverse) slope of the representative agent’s demand curve, captures the willingness of the market as a whole to accommodate trade at prevailing prices. Even in a frictionless equilibrium, this willingness is not unlimited as marginal trades lead to marginal changes in the risk characteristics of the representative agent’s holdings and hence to marginal changes in discount rates. Johnson (2006) develops the computation of this elasticity in general economies and demonstrates that it can have non-trivial dynamics, varying as the underlying state of the economy varies.

I study the properties of this measure in a simple two-asset economy in which one asset is money-like because holdings of it can be directly converted to current consumption. The second asset, which does not have this property, is just a claim to an endowment

³See especially Allen and Gale (2004), and Allen and Gale (2005) for a review.

stream that is meant to model the stock market. In this setting, agents in the economy can be said to be more or less liquid depending on the fraction of their assets that are technologically transformable from savings to consumption. I refer to these transformable holdings as liquid or (real) cash balances.

This is a standard buffer-stock savings model (Deaton (1991), Carroll (1992), Carroll (1997)), and a standard result is that the marginal propensity to consume out of available wealth rises with the percentage of cash holdings. This, in turn, implies that, faced with a marginal exchange of cash for risky shares, the representative agent will alter current consumption less when he has more cash, i.e. is more liquid. The intertemporal substitution aspect of trade is one feature that drives market illiquidity since altering current consumption changes discount rates. Hence, in this story, the market for shares of the second asset become endogenously more liquid when the economy as a whole is more liquid.

This explanation has the virtue of simplicity, relying as it does on elementary properties of well understood models. But that does not make it right. Financial constraints, credit channels, and segmented markets all may well contribute to the determination of market liquidity. The ideas are not mutually exclusive. However they may have very different implications about the role of institutions and the welfare effects of intervention. In particular, the results here imply that empirical evidence that there is *some* linkage between the two types of liquidity cannot necessarily be interpreted as evidence of incomplete contracts or inefficient markets.

Understanding liquidity dynamics is important for investors as well as policy makers. Liquidity risk is a topic of significant concern for any participant who may need to implement a dynamic portfolio strategy. Quantifying that risk, and possibly hedging it, requires an explicit model of the causes of liquidity fluctuations. To the extent that these factors are distinct from other, known sources of risk, investors may demand compensation for this risk, thus affecting equilibrium asset values. The model presented here constitutes an explicit and tractable quantification of time-varying liquidity, providing a complete description of its interaction with the underlying state variables.

The outline of the paper is as follows. In the next section, I introduce the economic setting. This is, in effect, the simplest economy one can study in which agents choose their relative holdings of liquid, consumable assets. I describe the economy formally and discuss equilibrium properties of consumption, savings, and asset prices. Of particular importance is the interaction between the consumption covariance of the risky dividend

stream and the level of liquid balances. In Section 3, I define the concept of market liquidity and show how to compute it in this model. I analyze the determinants of this quantity and highlight the intuition behind them. The primary result is that market liquidity increases with the level of cash holdings. While analytical proofs are mostly unavailable, the interaction is illustrated, and the intuition developed, through numerical examples using standard parameter values. The final section summarizes the paper’s contribution and concludes by highlighting the distinguishing empirical implications of the model.

2. An Economy with Liquid Balances

The balance sheet of a firm or an individual is said to be liquid when cash constitutes a high percentage of investments. In this context, “cash” refers to any money-like securities whose defining property is their ability to be exchanged *directly* for needed goods and services, without needing to rely upon the existence of (or conditions in) a secondary market to convert them to another medium of trade. This is a real property of an investment, which is equivalent to the concept of reversibility. The contrast with other assets, whether equipment or intellectual property, is that, even if perfect markets for claims to these exist, the capital stock committed to them may not itself be transformable to other uses. This section develops a standard model in which agents choose how much of their wealth to hold in liquid, convertible form. This percentage of liquid assets then becomes the main (endogenous) state variable driving consumption and asset prices. The aim at present is to study the interrelationships this implies between the economy’s liquidity and the other characteristics of the equilibrium.

2.1 Basic Properties

The setting is as follows. Time is discrete and an infinitely-lived representative agent has constant relative risk aversion (CRRA) preferences over consumption of a single good. The agent receives a risky stream, D_t , of that good in each period from an endowment asset. In addition, the agent has access to a second investment technology whose capital stock can be altered freely each period, and which returns a constant gross rate, $R \equiv e^{r \Delta t}$. For present purposes, the risklessness of the return is secondary. The primary feature of this investment is that, like cash, it can be drawn down or built up each period.⁴ By

⁴In terms of the canonical fruit tree metaphor, it is just preserved fruit.

contrast, the capital stock of the endowment asset can be neither increased nor decreased.

This is a version of the classic buffer-stock savings model (Deaton (1991), Carroll (1992)), a mainstay of the consumption literature. I use it here to investigate the behavior of the price of the endowment stream, which in that literature is interpreted as labor income, but which I interpret as dividends. The fact that claims to dividends may be tradeable (whereas labor income is not) in no way alters the equilibrium construction. Labor models also sometimes impose the constraint that investment in the savings technology must be positive, ruling out borrowing. This is intuitively sensible as a property of aggregate savings as well, but need not be imposed here, because it will hold endogenously anyway in the cases considered below.⁵ Moreover it is worth pointing out that there are also no financial constraints in the model. Agents in this economy may write any contracts and trade any claims with one another.

For parsimony, I have not included any other stochastic shocks to agent's supply of goods, such as labor income, which might be interpreted as a direct "liquidity shock". A natural way of doing so would be to make these proportional to dividends but with a transitory component. This specification would then look exactly like the process used for total risky income in the original buffer stock models (as in (Carroll 1997)). Separate idiosyncratic shocks are not needed, however, to deduce the effects of time-varying liquid balances on consumption and prices.

While the transformable investment was described as a technology, this asset could also be viewed as one-period riskless bonds (T-bills). Models with elastically supplied bonds are common in the asset pricing literature, but typically are used in conjunction with constant absolute risk aversion (CARA) utility functions. CARA utility is unsuitable here because it necessarily implies that demand for the risky asset is independent of the level of savings or liquid wealth. To my knowledge, the asset pricing implications of a riskless storage technology in a CRRA economy have not previously been analyzed.

Since consumption units carried over from one period to the next are perfectly fungible with current-period output, it is also natural to think of agents' total goods-on-hand as their "cash" balances, which can either be consumed or saved. They are like (commodity) money in that their consumption utility does not require any further transformation. Although the term "cash" is occasionally used below for these balances, it is worth clarifying that there is no fiat money in the economy. All quantities are real.

⁵With lognormal dividend shocks and CRRA preferences, agents will never borrow in a finite horizon economy. The policies below are limits of finite horizon solutions. When these limits exist and appropriate transversality conditions are satisfied, they are also solutions to the infinite horizon problem.

Since the aggregate stock of liquid savings is determined endogenously, there is no role for a government sector in the model. Nevertheless, one can still consider the effect of policy actions that alter the real quantity of liquid balances. Central bank interventions that change the net supply of bank deposits (i.e the monetary base) could be an example of such an action. I return to this topic in Section 2.3 below.

Setting the notation, the investor's problem is to choose a consumption policy, C_t , to maximize

$$J_t = \mathbb{E}_t \left[\sum_{k=1}^{\infty} \beta^k \frac{C_{t+k}^{1-\gamma}}{1-\gamma} \right],$$

where $\beta \equiv e^{-\phi}$ is the subjective discount factor, and I have set the time interval to unity for simplicity. A key variable is the total amount of goods, G_t , that the agent could consume at time t , which is equal to the stock of savings carried into the period plus new dividends received:

$$G_t = R(G_{t-1} - C_{t-1}) + D_t.$$

The end-of-period stock of goods invested in the transformable asset will be denoted $B_t \equiv G_t - C_t$. It is also useful to define the "income" received each period $I_t \equiv (R - 1)B_{t-1} + D_t = (R - 1)(G_t - D_t)/R + D_t$. Although this quantity plays no direct role, it helps in understanding savings decisions.

To take the simplest stochastic specification, I assume D_t is a geometric random walk:

$$D_{t+1} = D_t \tilde{R}_{t+1}, \quad \log \tilde{R}_{t+1} \sim \mathcal{N}(\mu - \frac{\sigma^2}{2}, \sigma^2)$$

so that dividend growth is *i.i.d.* with mean e^μ . Including a transient component (as would be appropriate in labor models) will not alter the features of the model under consideration here. As it is, the model is defined by five parameters: R , β , γ , μ , and σ .

The state of the economy is characterized by G_t and D_t which together determine the relative value of the endowment stream. It is easy to show the optimal policy must be homogeneous of degree one in either variable. So it is convenient to define

$$v_t = D_t/G_t$$

which takes values in $(0, 1)$. This variable can also be viewed as summarizing the real liquidity of the economy. As $v_t \rightarrow 0$, the endowment stream becomes irrelevant and all the economy's wealth is transformable to consumption whenever desired, and income fluctuations can be easily smoothed. As $v_t \rightarrow 1$ on the other hand, all income effectively

comes from the endowment asset whose capital stock cannot be adjusted. Since cash holdings are small, agents have little ability to dampen income shocks.

The agent's decision problem is to choose how much of his available goods to consume at each point in time. Intuition would suggest that he will optimally consume less of the total, G_t , when v is low than when v is high, since, in the former case consuming the goods amounts to eating the capital base, whereas in the latter case, G_t is mostly made up of the income stream, D_t , which can be consumed with no sacrifice of future dividends. This property is, in fact, true very generally.⁶

Proposition 2.1 *Assume the infinite-horizon problem has a solution policy $h \equiv C/G = h(v)$ such that $h(v) < 1$. Then,*

$$h' > 0.$$

Note: all proofs appear in Appendix A.

This result can also be understood by noting that $\partial C(D, G)/\partial D = h'$ so that the assertion is only that consumption increases with (risky) income, which is unsurprising when shocks to D are permanent. Moreover, the result holds for much more general preferences. This follows from the results of Carroll and Kimball (1996) who show that $\partial^2 C/\partial G^2 < 0$ whenever $u''' u'/[u'']^2 > 0$. Concavity implies that $0 < C - G \partial C/\partial G$ and the latter quantity also equals h' .

That h rises with v is essentially the only feature of the model that is necessary for the subsequent results. However further useful intuition about the dynamics of the model can be gained by considering how consumption behaves at the extreme ranges of the state variable v_t .

In the limit as $v_t \rightarrow 1$ the economy would collapse to a pure endowment one (Lucas 1978) if the agent consumed all his dividends, i.e. if $h(1) = 1$. This will not happen if the riskless savings rate available exceeds what it would be in that economy. That is, if

$$\frac{1}{R} < \mathbb{E}_t \left[\beta \left[\frac{C_{t+1}}{C_t} \right]^{-\gamma} \mid h(1) = 1 \right] = \mathbb{E}_t \left[\beta \left[\frac{G_{t+1} h(v_{t+1})}{G_t h(1)} \right]^{-\gamma} \mid h(1) = 1 \right] = \mathbb{E}_t \left[\beta \left[\tilde{R}_{t+1} \right]^{-\gamma} \right] \quad (1)$$

then the agent's marginal valuation of a one-period riskless investment, assuming no savings, exceeds the cost of such an investment. So no savings cannot be an equilibrium,

⁶Indeed, even the hypothesis $h < 1$ in the proposition is not necessary. The proof of the more general case is cumbersome, and is omitted.

and the conclusion is that the inequality (1) implies $h(1) < 1$. In the lognormal case, this is equivalent to $r > \phi + \gamma\mu - \gamma(1 + \gamma)\sigma^2/2$, the right-hand expression being the familiar interest rate in the Lucas (1978) economy.⁷

As dividends get small relative to cash, $v_t \rightarrow 0$, the economy begins to look like one with only the riskless asset. For such an economy, it is straightforward to show that the optimal consumption fraction is $h_0 \equiv (R - (R\beta)^{1/\gamma})/R$. Now if the agent's consumption fraction approached this limit (which it does, $h(\cdot)$ is continuous at zero⁸), his cash balances would grow at rate approaching $(R\beta)^{1/\gamma}$. If this rate exceeds the rate of dividend growth then v_t will shrink further. However, in the opposite case, the agent is dissaving sufficiently fast to allow dividends to catch up. Hence v_t will tend to rebound. Thus the more interesting case is when

$$(R\beta)^{1/\gamma} < E_t [\tilde{R}_{t+1}] \quad (2)$$

or $r < \phi + \gamma\mu$ under lognormality. A somewhat stronger condition would be that the agent actually dissaves as $v_t \rightarrow 0$. This would mean consumption $G_t h_0$ exceeds income $(R - 1)(G_t - D_t)/R + D_t$ or, at $v = 0$, $h_0 < (R - 1)/R$, which implies $(R\beta)^{1/\gamma} < 1$ or simply $r < \phi$.

With (1) and (2), then, the state variable v_t is mean-reverting.⁹ This requirement is not necessary for any of the results on prices or liquidity. However it makes the equilibrium richer. As an illustration, I solve the model for the parameter values shown in Table 1.

The parameters are fairly typical of calibrations of aggregate models when the endowment stream is taken to be aggregate dividends (with R approximating the real interest rate), and the preference parameters are all in the region usually considered plausible. Figure 1 plots the solution for the consumption function for these parameters. Analytical solutions are not available. However, as the proposition above indicated, h is increasing and concave. Also plotted is income (as a fraction of G), which is $(R - 1 + v)/R$. This

⁷The opposite inequality to (1) is sometimes imposed in the buffer stock literature to ensure dissavings as $v_t \rightarrow 0$. In that case, the specification of D_t is altered to include a positive probability that $D_t = 0$ each period. This assures the agent will never put $h = 1$.

⁸This is shown in Carroll (2004) under slightly different conditions. Modification of his argument to the present model is straightforward.

⁹Stationarity of v implies stationarity of consumption as a fraction of available wealth $C/G = h(v)$. Both are general properties under the model of Caballero (1990) who considers CARA preferences. Clarida (1987) provides sufficient conditions under CRRA preferences when dividends are *i.i.d.* Szeidl (2002) generalizes these results to include permanent shocks. No assumption about the long-run properties of the model are used below.

Table 1: *Baseline Parameters.*

PARAMETER	NOTATION	VALUE
coefficient of relative risk aversion	γ	6
subjective discount rate	$\phi = -\log \beta$	0.05
return to cash	$r = \log R$	0.02
dividend growth rate	μ	0.04
dividend volatility	σ	0.14
time interval	Δt	1 year

shows the savings behavior described above: for small values of v agents dissave, whereas they accumulate balances whenever v exceeds about 0.12.

The graph also indicates that, away from the origin the function h is actually quite close to linear in v . Numerical experimentation indicates that this is a robust qualitative feature of solutions. With it, some of the dynamic properties of the endogenous state variables become easier to understand.

Analysis of the variable $w_t \equiv (G_t - D_t)/D_t = RB_{t-1}/D_t$ is more tractable than of v_t . This is the ratio of beginning-of-period cash balances (i.e. before dividends and consumption) to dividends, and the two are related by $w = \frac{1}{v} - 1$. So the consumption function $h(v)$ can be expressed equivalently as $h(w)$. Then we can write w_{t+1} in terms of time- t quantities as follows:

$$w_{t+1} = \frac{R([G_t - D_t] + D_t - C_t)}{D_t \tilde{R}_{t+1}} = \frac{R}{\tilde{R}_{t+1}}(1 - h(w_t))(w_t + 1).$$

Let the random ratio R/\tilde{R} define a new mean-zero variable \tilde{Z} :

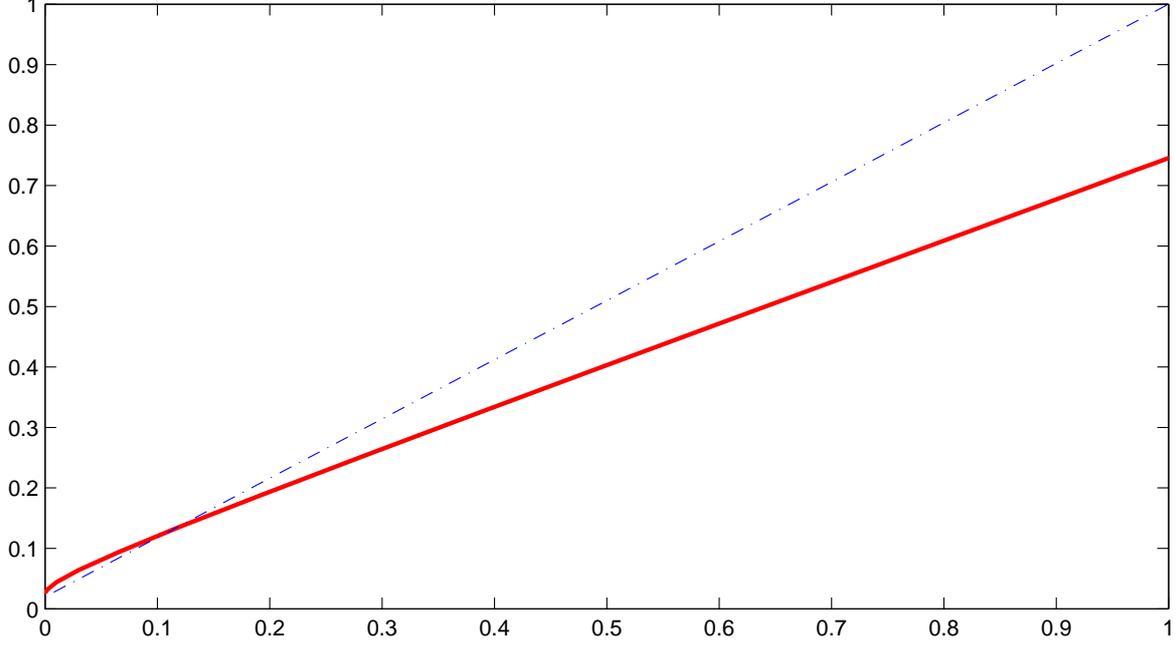
$$\frac{R}{\tilde{R}_{t+1}} \equiv \tilde{Z}_{t+1} + \bar{Z}$$

with $\bar{Z} \equiv E(R/\tilde{R})$ which is $e^{(r-\mu)}$. Subtracting w_t from w_{t+1} gives

$$\Delta w_{t+1} = \bar{Z}[(1 - h(w_t))(w_t + 1)] - w_t + [(1 - h(w_t))(w_t + 1)] \tilde{Z}_{t+1}. \quad (3)$$

This expression isolates the first and second moments of the w innovations. Their forms become clearer if we invoke the linear approximation for the function h . Specifically,

Figure 1: *Optimal Consumption*



The dark line is the optimal consumption function $\check{h}(v) \equiv C/G$ plotted against the ratio of dividends to total goods-on-hand $v \equiv D/G$. Also shown is income as a fraction of G plotted as a dashed line. All parameter settings are as in Table 1.

$$h(v) \doteq h_0 + (h_1 - h_0)v \Rightarrow h(w) \doteq \frac{h_0 w + h_1}{w + 1}.$$

Plugging this expression into equation (3) gives

$$\begin{aligned} \Delta w_{t+1} &\doteq \left(\bar{Z}[(1 - h_1) + (1 - h_0)w_t] - w_t \right) + [(1 - h_1) + (1 - h_0)w_t] \tilde{Z}_{t+1}. \\ &= \left(\bar{Z}(1 - h_1) - [1 - \bar{Z}(1 - h_0)]w_t \right) + [(1 - h_1) + (1 - h_0)w_t] \tilde{Z}_{t+1}. \end{aligned} \quad (4)$$

Here we see that w_t is approximately an affine process. The coefficient on $-w_t$ in the deterministic part of this specification can be interpreted as the speed of mean reversion. Using the expression given above for h_0 and the definition of \bar{Z} , the coefficient becomes

$$1 - (R\beta)^{1/\gamma} E(1/\tilde{R}) = 1 - e^{\frac{(r-\phi)}{\gamma} - \mu} \doteq \mu - \frac{(r - \phi)}{\gamma}.$$

This shows that the inequality (2) imposed earlier also implies (up to an approximation) that the process w_t mean-reverts, and, indeed, characterizes the degree of mean reversion

by the degree to which that inequality holds. Taking the parameter values given above, the coefficient in the expression evaluates to 0.045 which corresponds to a characteristic time scale (or half-life) of 15.4 years. In this economy, then, this is the “business cycle frequency” which governs the endogenous changes in liquidity, which, in turn, determine the consumption and saving behavior.

Having (approximately) characterized the dynamics of the state variable w_t (and hence v_t), we can directly infer the consumption dynamics by again appealing to the linear approximation of $h(\cdot)$. Since

$$C_t/D_t = h(v_t)/v_t = h(w_t)(w_t + 1) \doteq h_0 w_t + h_1,$$

consumption growth has two sources. The first is simply the growth rate of dividends, which is *i.i.d.* The second is the growth in the linear transformation of w_t which itself is an affine process. This second term’s expected growth will be high when w_t is itself low or when v is high. Intuitively, when v is high, current consumption is less than income. But since v is expected to mean revert, agents expect to be saving less in the future. That is, their consumption growth will benefit from the cyclical boost that will come from freer spending in more liquid times.

Consumption volatility varies in a similar, though more complicated, fashion. Dividend innovations are perfectly negatively correlated with the shocks \tilde{Z} driving w . So the two components of consumption growth work in opposite directions. And, from equation (4), the component $h_0 w_t + h_1$ will have percentage changes equal to

$$\frac{(1 - h_1) + (1 - h_0)w_t}{\frac{h_1}{h_0} + w_t}$$

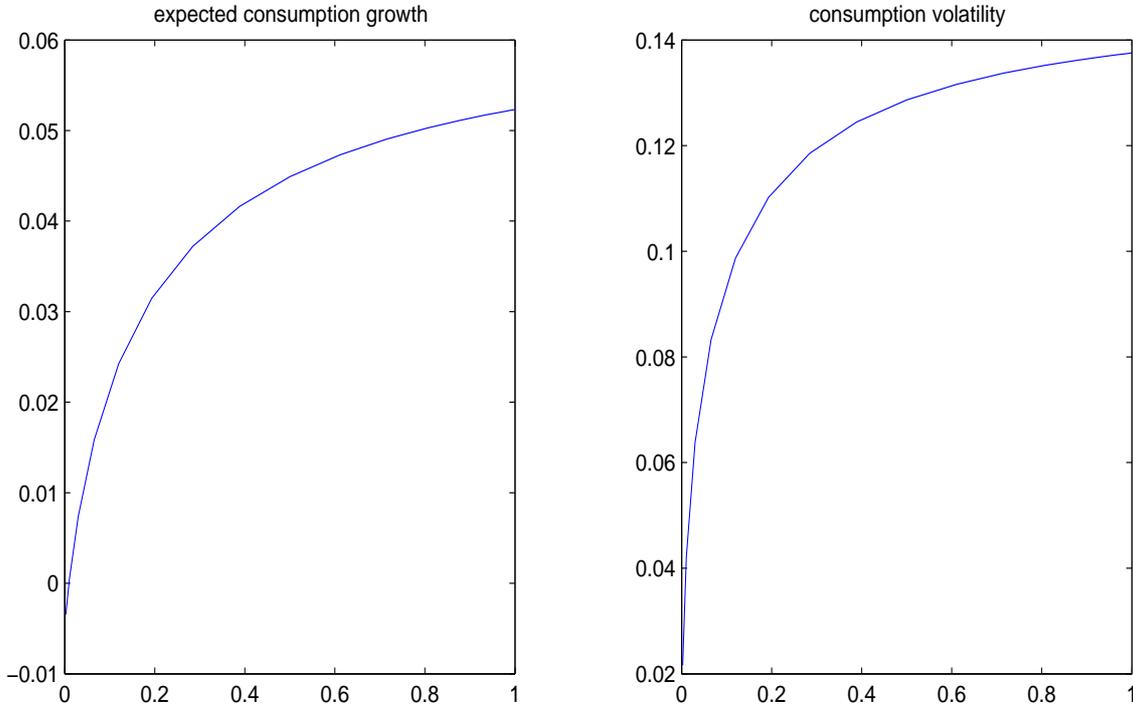
times those of \tilde{Z}_{t+1} . This fraction is increasing in w and typically less than unity.¹⁰ It follows then that consumption volatility decreases as w increases (or as v decreases). While the mathematical analysis is somewhat opaque, the intuition is simple. The consumption stream is necessarily mostly made up of dividends when liquid wealth is low. And dividends are more volatile than cash.

To get a sense of the degree of variation of the consumption moments, I first evaluate them numerically for the baseline parameter values. The results are shown in Figure 2. Since agents smooth consumption, the standard deviation is below that of the dividends

¹⁰As $w \rightarrow \infty$ it goes to $(1 - h_0)R = (R\beta)^{1/\gamma}$ which could exceed unity. In that case, positive dividend shocks would lower consumption.

themselves. More notably, the moments vary significantly as the liquidity of the economy varies.

Figure 2: *Consumption Moments*

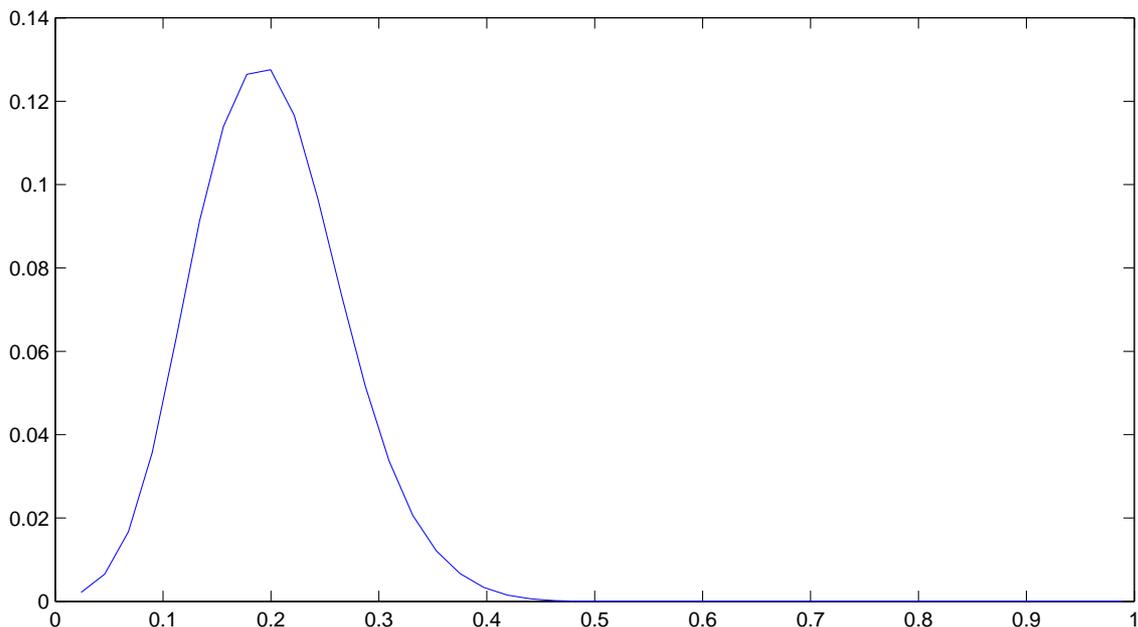


The left panel plots the conditional mean of log consumption growth, and the right hand panel plots the conditional standard deviation. The horizontal axis is v , the liquid balances ratio. All parameter settings are as in Table 1.

But how much does that liquidity vary? Figure 3 shows the unconditional distribution of v , calculated by time-series simulation. Its mean and standard deviation are 0.204 and 0.065 respectively, implying a plus-or-minus one standard deviation interval of (0.139, 0.269). Using this distribution to integrate the conditional consumption moments, the unconditional mean and volatility of the consumption process are 0.031 and 0.110. In terms of dynamic variation, the standard deviation of the conditional mean and volatility are 0.0051 and 0.0079, respectively.

To recap, this subsection has shown some important, basic properties of consumption in this model. The propensity to consume current goods, $h()$, is increasing in the liquidity ratio, v , the percentage of current wealth coming from dividends. Subject to some parameter restrictions, this percentage (or equivalently w) is stationary, with a degree of mean-reversion determined by μ , r , ϕ , and γ . The consumption ratio, h , and consumption growth are then also stationary with the same characteristic time-scale. Remarkably,

Figure 3: *Unconditional Distribution of Dividend-Cash Ratio*



The figure shows the unconditional distribution of v , the dividend-to-cash ratio, as computed from a 40,000 realizations of a time-series simulation. The first 500 observations discarded and a Gaussian kernel smoother has been applied. All parameter settings are as in Table 1.

although the exogenous environment is *i.i.d.*, states with high and low levels of liquidity (or accumulated savings) seem very different. When liquidity is low, consumption is more volatile and income is saved; when liquidity is high, agents dissave and, though expected consumption growth is lower, it is smoother. These consumption dynamics lead to the main intuition needed to understand asset pricing in this economy.

2.2 Asset Prices

Having seen that consumption is less volatile when liquid balances are high, one can immediately infer that discount rates will be lower in these states since marginal utility is smoother and hence the economy is less risky. Not only is this true, but a second factor reinforces this conclusion in terms of the risk of the stock market. When liquid balances are high, dividends also make up a lower fraction of consumption, hence, mechanically, the correlation of dividends with consumption is lower. Thus a claim on the dividend stream has less fundamental exposure when G is high relative to D .

Analytical expressions are again not attainable, and one must compute asset prices numerically from the usual first-order condition. With the current notation, and in terms

of the price-dividend ratio $g = g(v) \equiv P/D$, this condition is

$$g(v_t) = \beta \mathbb{E}_t \left[\left(\frac{v_t h(v_{t+1})}{v_{t+1} h(v_t)} \tilde{R}_{t+1} \right)^{-\gamma} [1 + g(v_{t+1})] \tilde{R}_{t+1} \right].$$

The function $g(v)$ can then be found by iterating this mapping on the unit interval.¹¹ Once g is obtained, the distribution of excess returns to the claim can be evaluated from

$$\frac{(P_{t+1} + D_{t+1})}{P_t} = \tilde{R}_{t+1} [1 + g(v_{t+1})]/g(v_t).$$

Figure 4 plots both the price-dividend ratio, g , and the price-goods ratio, f , for the parameter values in Table 1. The first function affirms the intuition above that the dividend claim must be more valuable when v is lower. In fact, g becomes unbounded as v approaches zero. This is not troublesome however, because it increases slower than $1/v = G/D$. The plot of $f(v) = vg(v)$ goes to zero at the origin, indicating that the total value of the equity claim is not explosive. Indeed, f is monotonically increasing in v , which lend support to the interpretation of v as measuring the illiquidity of the economy, since this function is the ratio of the *value* of the non-transformable asset to the value of the transformable one.

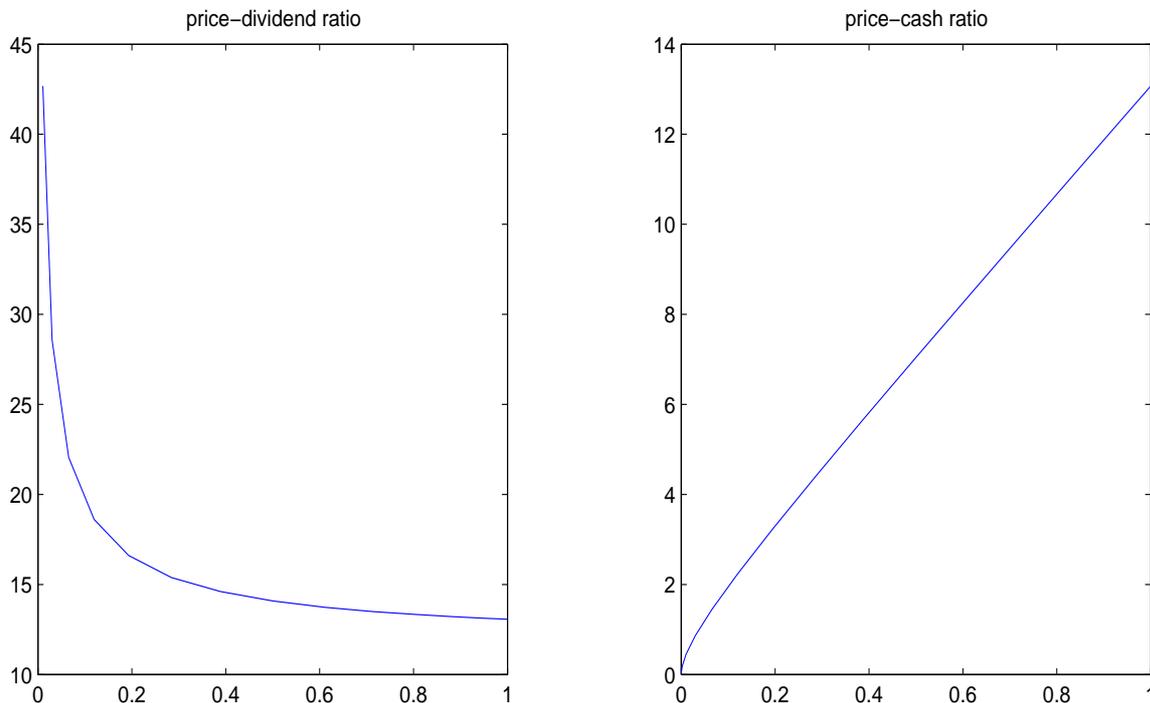
In light of the lack of closed form results, it is worth mentioning here the features of the pricing function that will matter below. Referring again to the figure, the fact that g explodes while f does not essentially bounds the convexity of g to be no greater than that of v^{-1} in the neighborhood of the origin. While not visually apparent, a similar convexity bound holds on the entire unit interval.¹² While the generality of this property is conjectural, extensive numerical experimentation suggests that it is robust. The curvature of the asset pricing function as a function of v is the key determinant of how much exogenous shocks to asset supplies will affect prices. As will be shown below in Section 3, that is tantamount to determining the liquidity of the securities market.

Figure 5 evaluates the expected excess returns and volatility as a function of v for the same parameter values. The plot verifies that expected returns are time-varying and predictable in this model, despite the constant dividend dynamics. When the economy is liquid, stock prices are high (as measured by the price-dividend ratio), expected excess

¹¹It is actually simpler to solve for the price-goods ratio $f(v) \equiv P/G = vg(v)$ which is not singular at the origin.

¹² Technically, the requirement is $g'' \leq 2(g')^2/g$ which holds for functions of the form $Av^{-\alpha}$ as long as $\alpha \leq 1$.

Figure 4: *Asset Pricing Functions*



The left panel plots the ratio $g \equiv P/D$, and the right hand panel plots P/G . The horizontal axis is v , the dividends-to-cash ratio. All parameter settings are as in Table 1.

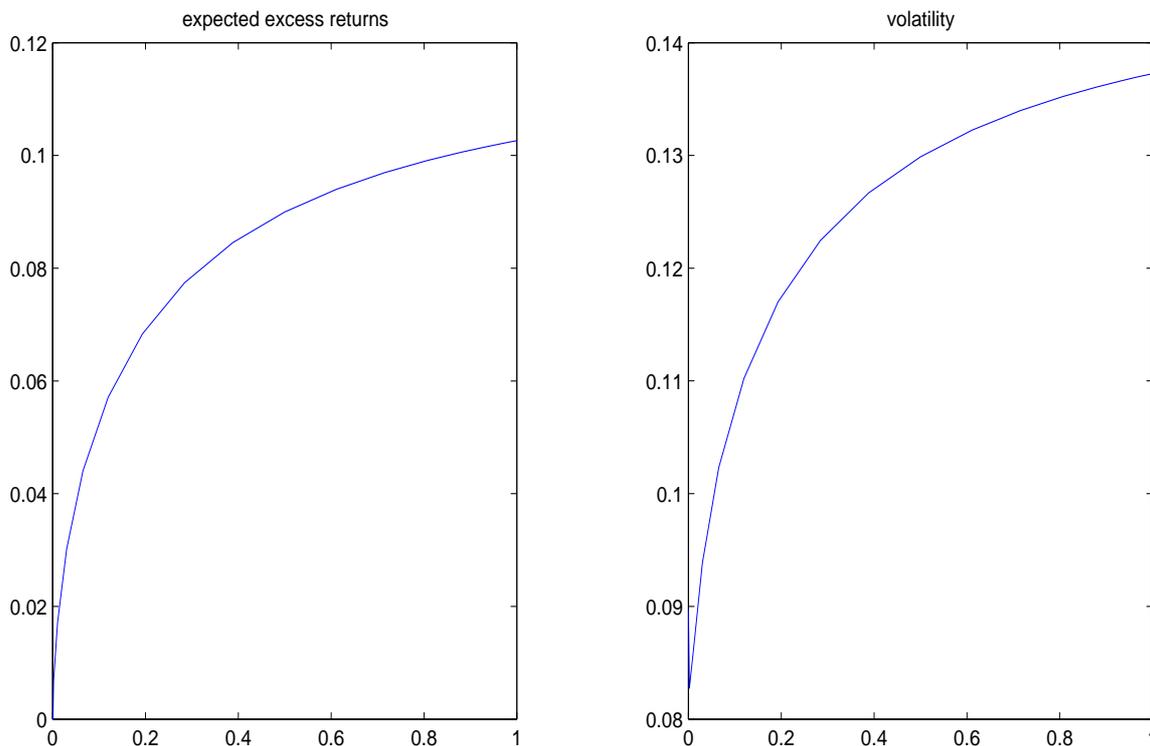
returns and volatility are low. The model thus offers a rich theory of time varying moments, which seems in accordance with the empirical facts about aggregate predictability. Intriguingly, it points to a new state variable, the economy's overall real liquidity, as a driving factor behind the asset moment dynamics.

2.3 Intervention

Before addressing stock market liquidity, it is useful to extend the model by thinking about how the equilibrium would be affected by changes in the relative supplies of the assets. In particular, consider an intervention (e.g. by the central bank) aimed at adjusting the economy's real liquidity. Interestingly, such interventions – subject to a simple condition – can be easily incorporated in the model without altering the savings or pricing laws.

Specifically, let today be t and assume that at some $\tau > t$ a random process will dictate a positive quantity ΔG to be added to the representative agent's cash holdings, G_τ , in exchange for a number of shares $\Delta X^{(1)}$ of the endowment stream, to be determined so

Figure 5: *Asset Return Moments*



The left panel plots the conditional mean of continuously compounded excess returns, and the right hand panel plots their conditional standard deviation. The horizontal axis is v , the dividend-cash ratio. All parameter settings are as in Table 1.

that the agent is indifferent to the exchange, i.e. it leaves his value function unchanged. In other words, the central bank engages in an open-market transaction at the competitive market price. What would such an intervention accomplish?

From a comparative static point of view, the answer is immediate. A purchase by the central bank simply shifts the ratio v to the left, as the numerator decreases and the denominator increases. To be careful, the previous notation needs to be augmented to reflect the variable number of shares of the endowment claim (heretofore implicitly set to one). So write

$$v_t \equiv \frac{D_t}{G_t} = \frac{D_t^{(1)} X_t^{(1)}}{G_t}.$$

That is, the superscript will denote per-share quantities. Thus, also, $P^{(1)} = D^{(1)} g(v)$ will be the per-share price of an endowment claim. If the representative agent sells shares, then, his stream of dividends is lowered, which is what v measures. The per-share dynamics of the D process is not changed however.

Thus, other than the perturbation to v_t , the economy is unaltered by the intervention.

Its effect therefore, according to the analysis above, would be to increase the price-dividend ratio and lower the risk premium, as well as to increase consumption relative to income and to reduce the volatility of consumption and of the stock market. This is perhaps a surprising result: a feasible intervention (i.e. involving no net transfer of wealth, by assumption) succeeds in altering the real economy in a non-trivial, and seemingly desirable, manner.

Is this conclusion justified from a dynamic point of view? Or would rational anticipation of the intervention at τ alter the equilibrium at t , rendering the comparative statics invalid? As the following proposition shows, the analysis is actually robust to interventions quite generally.

Proposition 2.2 *Let $\{\tau_k\}_{k=1}^K$ be an increasing sequence of stopping times $t < \tau_1 \dots \tau_K$, and let $\{\delta_k\}_{k=1}^K$ be a sequence of random variables on \mathcal{R}^+ . Suppose that at each stopping time an amount $\Delta G_{\tau_k} = G_{\tau_k} (\delta_k - 1)$ of goods are added to the representative agent's holdings in exchange for an amount of shares $\Delta X_k^{(1)}$ that leaves his value function unchanged. (If no such quantity exists, no exchange takes place.) Then, the value function, $J = J(v, D)$, consumption function, $h(v)$, and pricing function, $P(v, D)$, at time t are identical functions to those in the economy with no interventions when the endowments are fixed at their time- t amounts.*

The underlying logic of the proposition is simple: since the agent knows the intervention won't alter his value function at the time it occurs, the *ex ante* probability distribution of future value functions is unchanged. Hence today's optimal policies are still optimal, regardless of the intervention, which means the value function today is unaltered.

The key assumption, that the exchanges are value-neutral, is equivalent to imagining that decentralized agents compete perfectly in an auction for the shares and that the intervening entity then acts as a price taker. Implicitly, then, this entity is viewed as possessing real assets (or capable of creating them) prior to the intervention, but as not participating in the economy otherwise. Finally, although the stochastic nature of the interventions is essentially unrestricted in the propositions, there is an implicit assumption that the realization of the random variables does not alter agent's information set by conveying information about future values of D . Not ruled out, are intervention amounts and times that depend on the current state of the economy.

To summarize, then, the liquidity of the economy, defined as the ratio of real balances to non-transformable assets, can evolve via two different mechanism. First, exogenous

dividend shocks and endogenous savings decisions drive the ratio v higher or lower every period. Second, discrete interventions can periodically re-start the stochastic evolution from a new point. One might naturally view the central bank as intervening to add real liquidity to the economy when it gets “too low”, thus altering the stationary distribution of v , or even enforcing a stationary distribution if the parameter values did not ensure that v would otherwise have one.¹³

This completes the depiction of a simple economy in which there is a real savings technology and a single source of randomness. In this setting, the percentage of liquid assets is an endogenous state variable that determines all relative prices. Interestingly, although the agent faces the exact same external uncertainty in all states (dividend growth is *i.i.d.*), high liquidity and low liquidity states *seem* quite different. In particular, when liquid balances are low, asset prices are low relative to fundamentals and both consumption and asset prices are volatile.

3 Changes in Market Liquidity

In what sense can financial claims in the economy described above be said to be illiquid? After all, no actual trade in such claims takes place in the model, and, if it did, there are no frictions to make transactions costly.

Nevertheless, these observations do not mean that the market’s demand curve for risky securities is flat. In fact, in general, this will not be the case: marginal perturbations to a representative agent’s portfolio will marginally alter his discount rates, altering prices. This paper uses the magnitude of this price effect – essentially the slope of the representative agent’s demand curve – as the definition of a claim’s degree of illiquidity. It measures the price impact function that would be faced by an investor who did wish to trade with the market (i.e. with the representative agent) for whatever reason.¹⁴ Likewise, it measures the willingness of an agent (who has the holdings and preferences of the representative agent) to accommodate small perturbations to his portfolio.

Like the market price itself, this elasticity can be defined and computed whether

¹³Of course, in the context of the model, any such policy is irrelevant from a welfare standpoint since, by assumption, agents’ value functions are unaffected.

¹⁴There is a long history of price impact and trading cost models in the market microstructure literature. To my knowledge, Pagano (1989) was the first to identify the slope of the aggregate demand function – as opposed to the demand of an *ad hoc* market maker – as an equilibrium measure of liquidity.

or not the marginal perturbations actually take place in equilibrium. If the economy were disaggregated, and some subset of agents experienced idiosyncratic demand shocks forcing them to trade, they would incur trading costs as prices moved away from them in proportion to their required quantities. To the extent that these idiosyncratic demands do not alter aggregate risk and preferences, this liquidity trading may be regarded as going on in the background of any representative agent economy. The details of the trading needs do not effect the market illiquidity and need not be modeled explicitly.

Formally, the definition proposed in Johnson (2006) views the value and price functions of the representative agent as functions of his holdings, $X^{(0)}$ and $X^{(1)}$, of any two of the available assets. Illiquidity of asset one with respect to asset zero is then defined by the change in that agent's marginal valuation of asset one following a value-neutral exchange of the two assets, holding all other asset supplies fixed.

Definition 3.1 *The illiquidity $\mathcal{I} = \mathcal{I}^{(1,0)}$ of asset one with respect to asset zero is the elasticity*

$$\mathcal{I} = -\frac{X^{(1)}}{P^{(1)}} \frac{dP^{(1)}(\Theta(X^{(1)}), X^{(1)})}{dX^{(1)}} = -\frac{X^{(1)}}{P^{(1)}} \left(\frac{\partial P^{(1)}}{\partial X^{(1)}} - \frac{P^{(1)}}{P^{(0)}} \frac{\partial P^{(1)}}{\partial X^{(0)}} \right), \quad (1)$$

where $(\Theta(x), x)$ is the locus of endowment pairs satisfying $J(\Theta(x), x) = J(X^{(0)}, X^{(1)})$.

The definition stipulates that the derivative be computed along isoquants of the value function (parameterized by the curve Θ) and the second equality follows from the observation that value neutrality implies

$$\frac{d\Theta(X^{(1)})}{dX^{(1)}} = \frac{dX^{(0)}}{dX^{(1)}} = -\frac{P^{(1)}}{P^{(0)}}.$$

While the definition depends on the choice of asset zero, in many contexts there is an asset which it is natural to consider as the medium of exchange. In the model of Section 2 above, the storable asset is the obvious unit since its relative price in terms of goods is clearly constant, $P^{(0)} = 1$. This is, in fact, the property which justifies the interpretation of this asset as cash.¹⁵

The elasticity \mathcal{I} is a primitive endogenous quantity in any model. It answers the question: how much does the price move against someone for each share she trades. in

¹⁵Technically, one should distinguish between the quantity of claims to a unit of the physical asset and the capital stock, G , of that asset. But since the exchange rate is technologically fixed at unity, I make no distinctions below and use G and $X^{(0)}$ interchangeably.

$X^{(1)}$. Equivalently, it represents the percentage bid/ask spread (scaled by trade size and in units of $X^{(0)}$) that would be quoted by competitive agents in the economy were they required to make two-way prices. Thus, it captures familiar notions of illiquidity from the microstructure literature. Two simple examples can illustrate the theory, and also help clarify the computation in subsequent cases.

First, consider pricing a claim at time t to an asset whose sole payoff is at time T in a discrete-time CRRA economy. As usual, $P_t^{(1)} = E_t \left[\beta^{T-t} u'(C_T) D_T / u'(C_t) \right]$. Further suppose the asset is the sole source of time- T consumption: $C_T = D_T = D_T^{(1)} X^{(1)}$, and let the numeraire asset be any other claim not paying off at T or t . Then, differentiating,

$$\frac{dP_t^{(1)}}{dX^{(1)}} = \frac{1}{u'(C_t)} E_t \left[\beta^{T-t} u''(C_T) (D_T^{(1)})^2 \right] = -\gamma E_t \left[\beta^{T-t} \frac{u'(C_T)}{u'(C_t)} \frac{D_T^{(1)}}{X^{(1)}} \right] = -\gamma \frac{P^{(1)}}{X^{(1)}}$$

or $\mathcal{I} = \gamma$. In this case, the effect of asking the agent to substitute away from time- T consumption causes him to raise his marginal valuation of such consumption by the percentage γ , which is also the inverse elasticity of intertemporal substitution under CRRA preferences. If this elasticity were infinite, the claim would be perfectly liquid. Notice that the effect is not about risk bearing: no assumption is made in the calculation about the risk characteristics of the *other* asset involved. So the exchange could either increase or decrease the total risk of the portfolio.

Now consider a similar exchange of asset one for units of the consumption good. Then, in the computation of \mathcal{I} , there is an extra term in $dP^{(1)}/dX^{(1)}$ which is

$$E_t \left[\beta^{T-t} u'(C_T) D_T \right] \frac{d}{dX^{(1)}} \left(\frac{1}{u'(C_t)} \right) = -P^{(1)} \frac{u''(C_t)}{u'(C_t)} \frac{dC_t}{dX^{(1)}} = \gamma \frac{P^{(1)}}{C_t} \frac{dC_t}{dX^{(1)}}.$$

Now the value neutrality condition implies $dC_t/dX^{(1)} = -P^{(1)}$ so that \mathcal{I} becomes

$$\gamma \left[1 + \frac{P^{(1)} X^{(1)}}{C_t} \right].$$

It is easy to show that this is the illiquidity with respect to consumption for a general (i.e. not just one-period) consumption claim as well. Here the intertemporal substitution effect is amplified by a (non-negative) term equal to the percentage impact of the exchange on current consumption: $P^{(1)} X^{(1)} / C_t = \left| (X^{(1)} / C_t) dC_t / dX^{(1)} \right|$. This term may be either large or small depending on the relative value of future consumption. In a pure endowment economy with lognormal dividends and log utility, for example, current con-

sumption is $C_t = X^{(1)}D_t^{(1)}$ and the extra term is price dividend ratio, which is $1/(1 - \beta)$, which would be big. The intuition for this term is that marginally reducing current consumption (in exchange for shares) raises current marginal utility. So, if the representative agent is required to purchase $\Delta X^{(1)}$ shares and forego current consumption of $P^{(1)}\Delta X^{(1)}$, his discount rate rises (he wants to borrow) and he would pay strictly *less* than $P^{(1)}$ for the *next* $\Delta X^{(1)}$ shares offered to him.

In what follows, it will be useful to think of the mechanism in these examples as two separate liquidity effects. I will refer to that of the first example, captured by the term $\gamma \cdot 1$, as the future consumption effect, and that of the second, captured by $\gamma \cdot P^{(1)}X^{(1)}/C_t$, as the current consumption effect.

Returning to the model of Section 2, such explicit forms of \mathcal{I} in terms of primitives are not available. (Direct differentiation of the discounted sum of future dividends is intractable because future consumption depends in a complicated way on the current endowments.) However it is simple to express \mathcal{I} in terms of the functions g (the price-dividend ratio) and f (the value ratio), which are both functions of v , the dividend-liquid balances ratio.

Proposition 3.1 *In the model described in Section 2,*

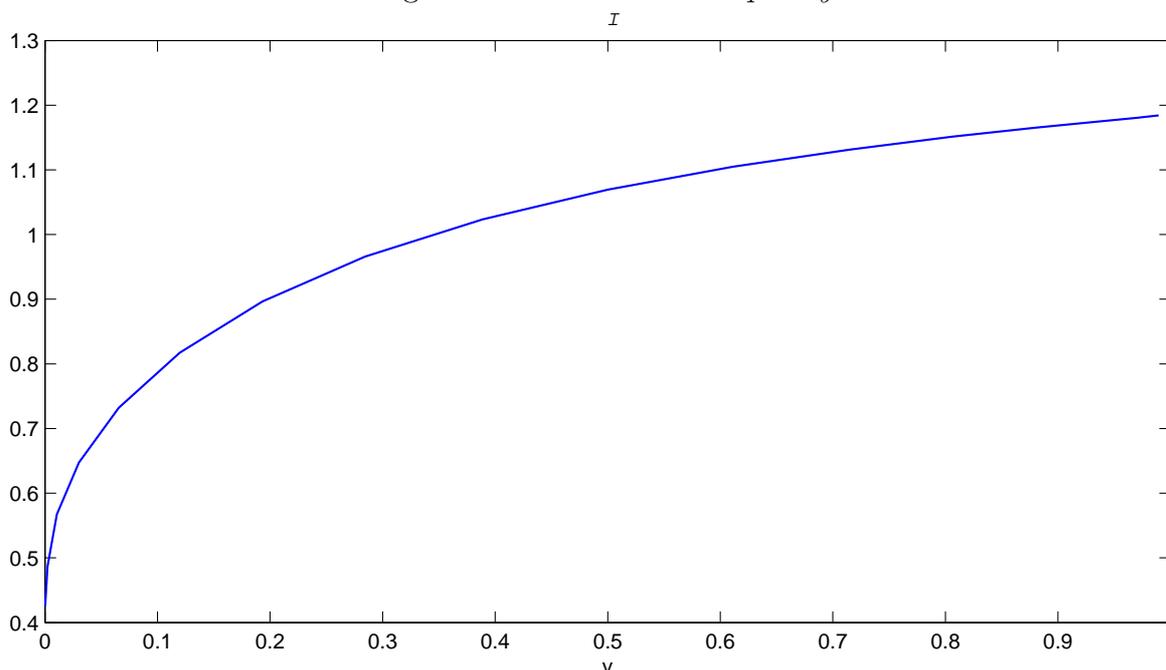
$$\mathcal{I} = -v(1 + vg(v))\frac{g'(v)}{g(v)} = (1 - v\frac{f'(v)}{f(v)})(1 + f(v)).$$

Illiquidity is positive in this economy because the price-dividend ratio is a declining function of v . Adding shares in exchange for cash mechanically shifts v to the right. As discussed above, when shares make up a larger fraction of the consumption stream their fundamental risk increases and their value declines.

Figure 6 plots illiquidity using the parameter values from Table 1. Notice first the most basic features, the level and variation of the function. The magnitude of illiquidity is both significant and economically reasonable. An elasticity of unity implies a one percent price impact for a trade of one percent of outstanding shares. This is the order of magnitude typically found in empirical studies of price pressure for stocks. Further, market liquidity is time-varying in this model. It is not a distinct state variable, of course, yet it is still risky in the sense of being subject to unpredictable shocks. While the current parameters restrict v , and hence \mathcal{I} , to a rather narrow range, even so, it is possible for illiquidity to more than double.¹⁶

¹⁶Incorporating separate transitory “liquidity shocks”, as discussed above, would broaden the stochas-

Figure 6: *Stock Market Illiquidity*



The figure shows the elasticity \mathcal{I} as a function of v for the model of Section 2. All parameter settings are as in Table 1.

This brings us to the topic of how and why market liquidity changes here. The figure clearly provides the fundamental answer: illiquidity rises when v does. Or, to stress the main point, the stock market is more liquidity when liquid assets are in greater relative supply. This is the heart of the paper's results.

To understand why this occurs, consider the role that the availability of a savings technology plays in the determination of price impact. In effect, it dampens both of the illiquidity mechanisms in the pure-endowment examples above. When the liquidity provider (the representative agent) chooses to use some cash savings to purchase additional risky shares – instead of being forced to forego consumption – current marginal utility does not rise as much. In addition, marginally depleting savings today raises the expected marginal utility of future income, which raise the valuation of future dividends. Hence the impact of foregone consumption (illustrated in the earlier examples) on both the numerator and denominator of the marginal rate of substitution are buffered by the use of savings. But now recall from the last section that the propensity to save when given an extra unit of G (or to dissave when required to give up a unit) falls with v . Because h rises with v (for the very general reasons discussed previously), discount rates

tic range of v , increasing the variation in market liquidity.

are less affected by portfolio perturbations when v is low.

Appendix B analyzes the effects in more detail in the two-period case which corresponds to the previous examples. Even in this case, a formal proof that \mathcal{I} is increasing is unobtainable. However it is possible to isolate the individual terms in \mathcal{I}' and to see how each rises with the propensity to consume.

Mechanically, differentiating the expression in the proposition shows that \mathcal{I}' will be positive as long as $g(v)$ is not too convex. (See note 12.) In fact, it is sufficient that $\log g$ is concave. And concavity is equivalent to the assertion that g'/g gets bigger (more negative) as v increases, meaning the percentage price impact of an increase in v is increasing.

Summarizing, markets are illiquid in this economy ($\mathcal{I}(v) > 0$) because discount rates rise with the proportion of risky asset holdings. Market's are *increasingly* illiquid as cash balances decline ($\mathcal{I}'(v) > 0$) because this impact on discount rates itself rises. Discount rates are affected more strongly by portfolio perturbations when cash is low because consumption – not savings – absorbs a higher percentage of the adjustment.

Because the liquid balances ratio determines all dynamic quantities in this model, all the covariances of market liquidity immediately follow from the positive relation between \mathcal{I} and v . In particular, as the economy wanders into the low cash region, not only does it become more difficult to trade, but also stocks become cheap (the price dividend ratio falls) and risk premia and volatility go up.

In the stationary economy described by the baseline parameters in Table 1 the steady state distribution of v is not very diffuse, and the variations in liquidity are not particularly dramatic. But consider, instead, the parameter set shown in Table 2.

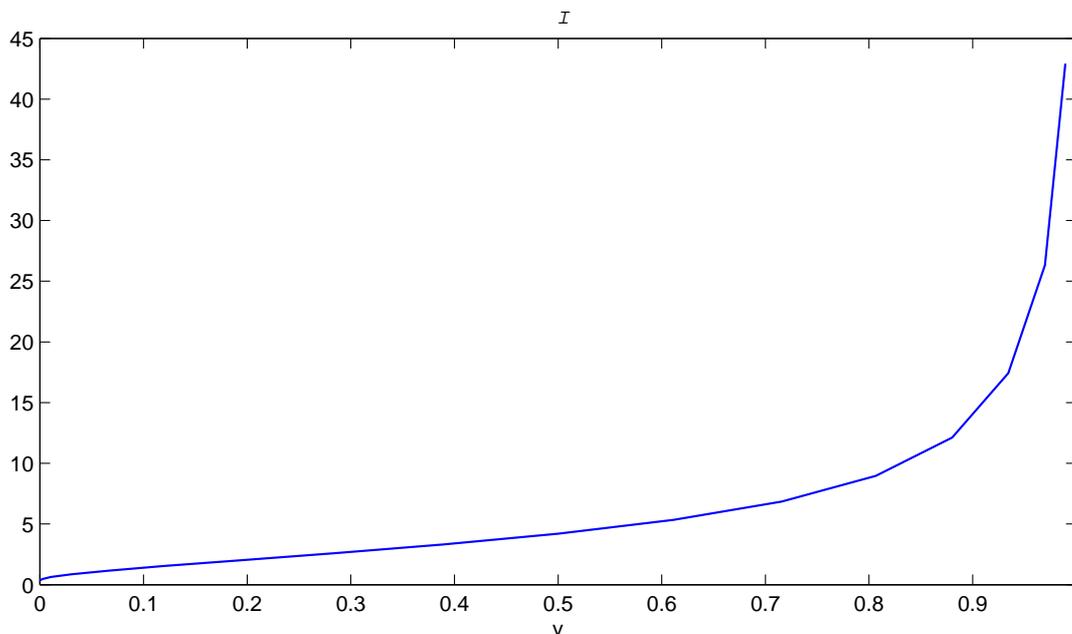
Table 2: *Nonstationary Model.*

PARAMETER	NOTATION	VALUE
coefficient of relative risk aversion	γ	2
subjective discount rate	$\phi = -\log \beta$	0.02
return to cash	$r = \log R$	0.02
dividend growth rate	μ	0.03
dividend volatility	σ	0.10

This version of the model has insufficient risk aversion to induce agents to save, even as their cash balances dwindle. As a consequence, the model is not stationary. This does

not present any problem for the computation of $\mathcal{I}(v)$, however. Figure 7 shows that, in fact, there is a dramatic deterioration in market liquidity with v for this economy, with $\mathcal{I}(v)$ rising by almost an order of magnitude between $v = 0.25$ and $v = 0.95$, which is approximately the range v will experience over 20 years.

Figure 7: *Stock Market Illiquidity: Nonstationary Economy*



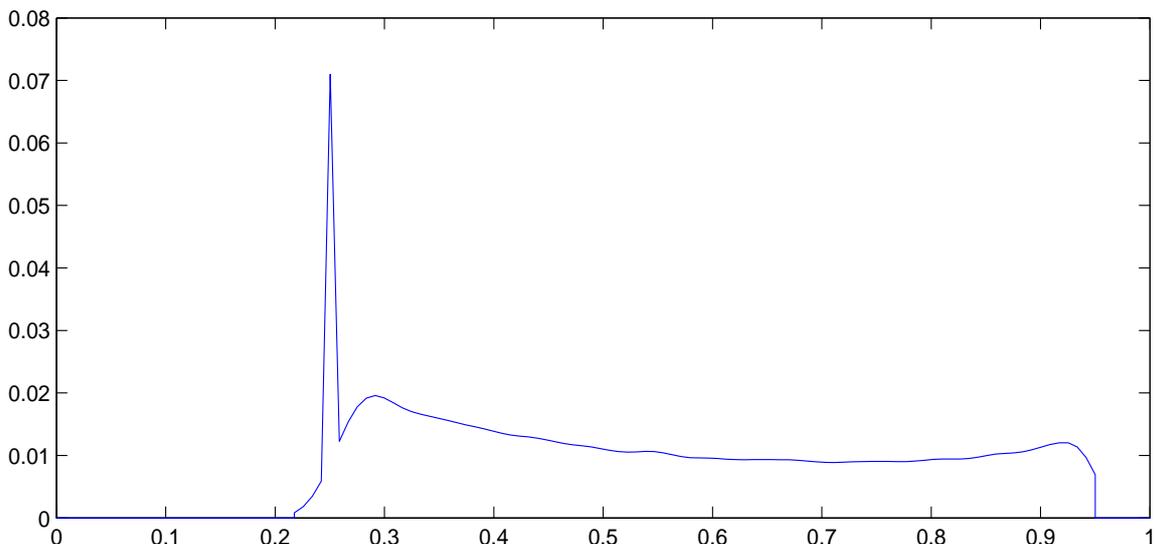
The figure shows the illiquidity, \mathcal{I} , as a function of v using the parameter settings shown in Table 2.

Recalling now the notion of intervention incorporated in the model, one could make this version effectively stationary by imagining a periodic “rescue” by the central bank when v approaches some higher limit. For example, suppose whenever v exceeds 0.95, an open-market operation is undertaken (as described in the last section) to re-start it at 0.25. This would induce the distribution shown in Figure 8. Here the injections of cash by the central bank in the extreme (high v) states would drastically improve market conditions.¹⁷ The severe sensitivity of prices to volume would be dampened, the price-dividend ratio would rise, and stock volatility would be quelled.

To an observer of this economy, it could well appear that the periodic deterioration of prices and increases in volatility occurred *because* of the lack of market liquidity. Moreover, the success of the intervention could seem to support the idea that the lack

¹⁷Proposition 2.2 justifies this interpretation because the computation of \mathcal{I} is not altered by the possibility of such interventions.

Figure 8: *Unconditional Distribution of Dividend-Cash Ratio*



The figure shows the unconditional distribution of v , the dividend-to-cash ratio, as computed from a 40,000 realizations of a time-series simulation. An intervention rule is applied, restarting the process at 0.25 whenever 0.95 is hit or exceeded. The first 500 observations discarded and a Gaussian kernel smoother has been applied. All parameter settings are as in Table 2.

of liquid assets (cash) caused a decline in intermediation, causing the rise in market liquidity, and leading to the seemingly distressed state.

The model here shows that neither of the above inferences need follow from the observed linkages. Market illiquidity and risk premia may rise simultaneously without the former having anything to do with the latter. A decrease in liquid balances can cause both, but without operating through the constraints of intermediaries. This is not to say that such constraints cannot have an amplifying effect on risk and risk premia, nor that understanding the details behind the operation of such constraints is not important for managing financial systems. However, at a minimum, one must be cautious in inferring that the degree of market illiquidity is a direct gauge of the importance of constraints.

4. Conclusion

The term “liquid” has (at least) two distinct meanings in finance, depending on whether it is applied to markets or to the asset holdings of an agent. A market is said to be liquid when it is easy to trade in the sense that buyers and sellers are readily

available, at similar prices, to accommodate transaction demand. An agent – a firm, a person, or an economy – is said to be liquid when a large fraction of its assets are readily convertible into tangible goods. This paper describes an economic mechanism linking the two concepts: when cash balances are low agents are less willing to accommodate others' trade demands because doing so entails more adjustment to current consumption. This is a primitive economic effect that is not driven by segmented markets, contracting frictions, information effects, details of microstructure, or irrationality.

I illustrated this theory in the context of a simple model incorporating savings decisions. Although the model is highly simplified (with only one exogenous source of risk), it provides a full description of liquidity determination in terms of economic primitives. Quantitative results show that market liquidity varies significantly with the level of liquid balances. Moreover the overall dynamic pattern of the economy seems right: when there is more cash around, the stock market is higher and less volatile, as well as more liquid.

Some of the theory's other implications are worth highlighting because they differ from those one would expect based on the intuition of the financial constraint type models discussed in the introduction.

First, under the present model, the two types of illiquidity would be always linked, not just in times of stress, i.e. when intermediaries' constraints bind. Second, the model suggests that, in explaining market conditions (including liquidity), the economy-wide level of real liquid balances is a key state variable. However, controlling for the financial position of the economy as a whole, there should be no role for the financial position of brokerage firms and other intermediaries in explaining aggregate fluctuations.

More broadly, and more controversially, the model makes the point that time-varying market liquidity – and even the occurrence of liquidity crises – need not be evidence of market failure or inefficiency. Even if intervention in credit markets is successful in promoting market stability, the mechanism may not be through alleviating financing frictions.

Appendix

A. Proofs

This appendix collects proofs of the results in the text.

Proposition 2.1

Proof. This proof will restrict attention to policy solutions in the class of limits of solutions to the equivalent finite-horizon problem. So consider the finite-horizon problem with terminal date T . Let h_t denote the optimal consumption-to-goods ratio at time t .

Clearly h_t cannot exceed one, since this would lead to a positive probability of infinitely negative utility at T . The assumption of the proposition is then that, at each v , we have an interior solution for h_t (at least for $T - t$ sufficiently large). In that case, h_t must satisfy the first order condition

$$h_t^{-\gamma} = R\beta E_t \left[\left[(R(1 - h_t) + v_t \tilde{R}_{t+1}) h_{t+1}(v_{t+1}) \right]^{-\gamma} \right]$$

where $v_{t+1} = v_t \tilde{R}_{t+1} / (R(1 - h_t) + v_t \tilde{R}_{t+1})$. I will assume a \mathcal{C}^1 solution exists for all t .

An implication of the first order condition is that the expectation

$$E_t \left[\left[\frac{h_t}{(R(1 - h_t) + v_t \tilde{R}_{t+1}) h_{t+1}(v_{t+1})} \right]^{+\gamma} \right] \quad (1)$$

must not be a function of v_t . I use this fact to prove the following successive properties:

- (i) $h'_t \leq h_t/v_t \quad \forall v_t, t$.
- (ii) $-(1 - h_t)/v_t \leq h'_t \quad \forall v_t, t$.
- (iii) $0 < h'_t \quad \forall v_t, t$.

For for the first point, assume the property holds for h_{t+1} but fails to hold for h_t . Write the expectation, above as

$$E_t \left[\left[\frac{h_t(v_t)}{\frac{h_{t+1}(v_{t+1})}{v_{t+1}}} \right]^{+\gamma} \tilde{R}_{t+1}^{-\gamma} \right]. \quad (2)$$

The hypothesis implies that the derivative with respect to v_t of the numerator in the inner brackets is positive and the derivative with respect to v_{t+1} of the denominator is negative. Also, the derivative $\frac{dv_{t+1}}{dv_t}$ is

$$\frac{R\tilde{R}_{t+1}}{(R(1-h_t) + v_t\tilde{R}_{t+1})^2} (1-h_t + v_th'_t).$$

The hypothesis on h_t implies that $(1-h_t + v_th'_t) \geq 1$. So $\frac{dv_{t+1}}{dv_t}$ is positive. Together, these observations imply that an increase in v_t will raise the numerator and lower the denominator of square bracket term in equation (2) for all values of the random variable \tilde{R}_{t+1} . Hence the expectation cannot be constant. The contradiction, combined with the fact that the final optimal policy is $h_T = 1$, which satisfies the induction hypothesis, proves $h'_t \leq h_t/v_t$ for all t .

Next, assume $-(1-h_{t+1})/v_{t+1} \leq h'_{t+1}$ but that the reverse holds for h_t . Differentiate the denominator of equation (1) to get

$$\frac{1}{(R(1-h_t) + v_t\tilde{R}_{t+1})} \left((\tilde{R}_{t+1} - Rh'_t)(R(1-h_t) + v_t\tilde{R}_{t+1}) h_{t+1} + \tilde{R}_{t+1} R (1-h_t + v_th'_t)h'_{t+1} \right)$$

Using the result just shown, $h'_{t+1} \leq h_{t+1}/v_{t+1}$. And, by the induction hypothesis, $(1-h_t + v_th'_t) < 0$. So the smallest the term in large parentheses can be is

$$\begin{aligned} & (R(1-h_t) + v_t\tilde{R}_{t+1}) h_{t+1}v_t^{-1} \left((\tilde{R}_{t+1} - Rh'_t)v_t + R(1-h_t + v_th'_t) \right) \\ & = (R(1-h_t) + v_t\tilde{R}_{t+1}) h_{t+1}v_t^{-1} \left(R(1-h_t) + v_t\tilde{R}_{t+1} \right) > 0. \end{aligned}$$

These observations imply that an increase in v_t will lower the numerator and raise the denominator of the bracketed term in (1) for all values of \tilde{R}_{t+1} . Hence the expectation cannot be constant. The contradiction, combined with the fact that the final optimal policy satisfies the induction hypothesis, proves $-(1-h_t)/v_t \leq h'_t$ for all t .

The third step proceeds similarly: assume the inequality (iii) holds for $t+1$ but not t . By the previous point (ii), we now have $(1-h_t + v_th'_t) \geq 0$ even though $h'_t < 0$. This means $\frac{dv_{t+1}}{dv_t}$ is always positive. So an increase in v_t must increase the denominator and decrease the numerator of (1), contradicting the constancy of the expectation. Given that $h_T = 1$, the constancy of the expectation at $T-1$ immediately implies that h_{T-1} must be strictly increasing. Hence h_{T-1} satisfies the induction hypothesis (iii). So we conclude $h'_t > 0$ for all $t < T$.

Now the limit of the discrete time maps: $h \equiv \lim_{t \rightarrow -\infty} h_t$ must also satisfy the condition that

$$\mathbb{E}_t \left[\left[\frac{h(v_t)}{(R(1-h) + v_t\tilde{R}_{t+1}) h(v_{t+1})} \right]^{+\gamma} \right]$$

is constant. The limit of increasing functions cannot be decreasing. However it can be flat. But

if $h(\cdot)$ is constant, then an increase in v_t would still raise $(R(1 - h_t) + v_t \tilde{R}_{t+1})$ and change the expectation. So we must also have $h' > 0$.

QED

Proposition 2.2

Proof. Let us distinguish, at each intervention date, between the times immediately before and immediately after the exchange, writing these as e.g., τ_k^- and τ_k^+ . (It is immaterial whether allocation and consumption decisions are made before or after.) Also write the value function of the representative agent as $J_t = J(D_t^{(1)}, X_t^{(1)}, G_t)$. Recall the superscript denotes per-share values, so that the total dividend income of the agent is $D_t = D_t^{(1)} X_t^{(1)}$.

Let J_t^o be the value function of the equivalent economy in which no further interventions will take place, that is, in which $X_s^{(1)} = X_t^{(1)}$ for all $s \geq t$, as in the original model. Similarly, let J_t^k be the value function under the assumption that the k th exchange does *not* take place. Then the assumption of the proposition that agents are indifferent to each exchange can be expressed as $J_{\tau_k^-}^k = J_{\tau_k^+}$.

Now consider the value function at any date t such that $\tau_{K-1} < t \leq \tau_K$. This must satisfy

$$J_t = \max_{\{C_{t+n}\}_{n=0}^{\infty}} E_t \left[\sum_{n=0}^{\infty} \beta^n u(C_{t+n}) \right].$$

The expectation can be written

$$\sum_{j=0}^{\infty} E_t \left[\sum_{n=0}^{j-1} \beta^n u(C_{t+n}) + \sum_{n=j}^{\infty} \beta^n u(C_{t+n}) | \tau_K = t + j \right] P(\tau_K = t + j)$$

and the inner, conditional expectation is

$$E_t \left[\sum_{n=0}^{j-1} \beta^n u(C_{t+n}) + E_{t+j} \left(\sum_{n=j}^{\infty} \beta^n u(C_{t+n}) | \tau_K = t + j \right) | \tau_K = t + j \right].$$

Define the conditionally optimal policy $\{C_{t+n}^j\}_{n=1}^{\infty}$ to be the one that maximizes this latter expectation. But, by assumption, the solution to

$$\max_{\{C_{t+n}\}_{n=j}^{\infty}} E_{t+j} \left(\sum_{n=j}^{\infty} \beta^n u(C_{t+n}) | \tau_K = t + j \right)$$

coincides with the same value in the absence of the exchange, which is

$$\max_{\{C_{t+n}\}_{n=j}^{\infty}} E_{t+j} \left(\sum_{n=j}^{\infty} \beta^n u(C_{t+n}) \mid \text{no trade at any date } s \geq t+j \right).$$

This solution is what was called J_{t+j}^K (times β^{j-1}).

Hence the conditionally optimal policy solves

$$\max_{\{C_{t+n}\}_{n=0}^{\infty}} E_t \left[\sum_{n=0}^{j-1} \beta^n u(C_{t+n}) \mid \text{no trade at any date } s < t+j \right] + \beta^{j-1} J_{t+j}^K$$

which is the same as J_t^K . Since this is true regardless of the conditioning index j , it follows that $J_t = J_t^K$ and also that the optimizing policies of the two problems coincide. Since K is the last exchange date, we also have that $J_t = J_t^o$ by definition of the latter.

We have shown that the value function and optimal policies are the same at $\tau_{K-1} < t \leq \tau_K$ as they would be if there were no intervention. By backward induction, the same argument applies at $\tau_{K-2} < t \leq \tau_{K-1}$, and so on. Since the optimal policies and value function are the same functions as in the no-intervention economy, it follows that the pricing function, and its derivatives must also be the same.

QED

Proposition 3.1

Proof. The elasticity \mathcal{I} can be computed from direct differentiation of $P^{(1)}$ in terms of f or g using the value-neutral condition $dG_t/dX^{(1)} = -P^{(1)}$ and

$$\frac{dv_t}{dX^{(1)}} = \frac{v_t}{X^{(1)}} (1 + v_t g(v_t)) = \frac{v_t}{X^{(1)}} (1 + f(v_t))$$

which follows from the definition $v = D^{(1)}X^{(1)}/G$.

QED

B. \mathcal{I} as a function of v : The Two-period Case.

This appendix analyzes the dependence of \mathcal{I} on v in a two-period CRRA economy analogous to the two-period pure-endowment example in Section 3. This illustrates how the availability of savings dampens both the current consumption effect and the future

consumption effect.

Suppose the representative agent has G_0 goods today and will receive a dividend XD_1 next period which he consumes and then dies. (I suppress the per-share superscript here to lighten the notation.) If the economy does not have a savings technology, so that $C_0 = G_0$, then, as in the example above, we have

$$\mathcal{I} = \gamma \left[1 + \frac{XP}{C_0} \right] = \gamma \left[1 + \frac{XP}{G_0} \right].$$

Now add the possibility of investing in cash and, for simplicity, fix the agent's savings to be $(1 - \lambda)G_0$ or $C_0 = \lambda G_0$ for some fraction λ . Ignore the determination of the optimal λ and view the price per share today as $P(\lambda)$. The algebra in calculating \mathcal{I} is a little messy, but worthwhile.

First,

$$P = P(G_0, X; \lambda) = \lambda^\gamma G_0^\gamma \beta \mathbb{E}[(XD_0 \tilde{R}_1 + (1 - \lambda)RG_0)^{-\gamma} D_0 \tilde{R}_1]$$

where I have written $D_1 = D_0 \tilde{R}_1$, as before. Differentiating this with respect to X subject to $dG_0/dX = -P$ and scaling by X/P produces three terms in \mathcal{I} .

$$\begin{aligned} \text{Term I :} & \quad \gamma D_0 X \lambda^\gamma G_0^{\gamma-1} \beta \mathbb{E}(XD_0 \tilde{R}_1 + (1 - \lambda)RG_0)^{-\gamma} \tilde{R}_1 \\ & = \gamma v_0 \lambda^\gamma \beta \mathbb{E}(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma} \tilde{R}_1 \\ \text{Term II :} & \quad -\gamma D_0 X \lambda^\gamma G_0^\gamma (1 - \lambda)R \beta \mathbb{E}(XD_0 \tilde{R}_1 + (1 - \lambda)RG_0)^{-\gamma-1} \tilde{R}_1 \\ & = -\gamma v_0 \lambda^\gamma (1 - \lambda)R \beta \mathbb{E}(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma-1} \tilde{R}_1 \\ \text{Term III :} & \quad \gamma D_0^2 X \lambda^\gamma G_0^\gamma \beta \mathbb{E}(XD_0 \tilde{R}_1 + (1 - \lambda)RG_0)^{-\gamma-1} \tilde{R}_1^2 / P \\ & = \gamma v_0 \frac{\mathbb{E}(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma-1} \tilde{R}_1^2}{\mathbb{E}(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma} \tilde{R}_1}. \end{aligned} \tag{3}$$

The first term can also be written as simply $\gamma XP(\lambda)/G_0$, which shows that it corresponds to the current consumption effect. But now it is also clear, from the second line, that this term declines as λ does. As the agent saves more, the impact of the value-neutral trade on current consumption is, of course, smaller.

The second term is another contribution from the effect of altering current consumption which arises because now (unlike in the endowment model) an increase in consumption goods today produces interest income next period. Comparing the second line with

the fourth, term II is strictly smaller than term I, because the only difference is an extra factor in the expectation of term II equal to $\Psi \equiv (1 - \lambda)R / (v_0 \tilde{R}_1 + (1 - \lambda)R) \leq 1$. Calling the term I integrand Γ , the two terms can also be combined, giving

$$\begin{aligned} & \gamma \frac{XP}{G_0} \left[1 - \frac{E\Psi \Gamma}{E\Gamma} \right] \\ &= \gamma \lambda^\gamma \beta E\Gamma\Theta = \gamma v_0^2 \lambda^\gamma \beta E(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma-1} \tilde{R}_1^2 \end{aligned}$$

where $\Theta \equiv (1 - \Psi) = v_0 \tilde{R}_1 / (v_0 \tilde{R}_1 + (1 - \lambda)R)$. And the last equation clearly still decreases as λ does, vanishing at $\lambda = 0$. As one would expect, the total impact of the current consumption terms is smaller when the current consumption ratio is lower.

Finally, term III is what I referred to as the future consumption effect in the earlier examples. It can be reexpressed as

$$\gamma \frac{E\Theta \Gamma}{E\Gamma} \leq \gamma.$$

Again, the ability to store goods lowers this term relative to the endowment economy cases. Intuitively, the presence of positive savings at date zero lowers the percentage impact of a change in dividends on future marginal utility. Somewhat less obviously, term III also decreases as λ does, regardless of the parameter values.¹⁸ Loosely, this is due to the extra term in the numerator, Θ , which behaves like $(1 - \lambda)^{-1}$.

Hence all the terms in \mathcal{I} are less than their counterparts in the corresponding endowment economy, and more so as λ declines. Now recall that the *optimal* consumption ratio is $\lambda = h(v)$ which is an increasing function of v . This shows the mechanism that causes the liquidity of the market for asset one to increase as the level of liquid balances in the economy does.¹⁹ The reason this happens is because the propensity to consume current wealth increases as liquid wealth declines, and this propensity, in turn, determines how big an impact a change in risky asset holdings has on marginal utilities. That impact dictates the willingness of agents to accommodate trades, or the rigidity of prices.

¹⁸This can be shown using the result that two monotonically related random variables must be positively correlated. The proof is available upon request.

¹⁹I have not proven, even in the two-period case, that $\mathcal{I}(v)$ must be increasing in v . The argument above does not consider either the variation of λ with v or the direct effect, i.e. not through the savings term, of v on \mathcal{I} . What the argument shows is that, which ever direction these other terms go, the savings channel always makes illiquidity rise with v .

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