

Preserving Bidder Privacy in Assignment Auctions: Design and Measurement

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Received: January 25, 2017

Revised: January 23, 2018; August 10, 2018; January 3, 2019

Accepted: March 10, 2019

Published Online in Articles in Advance:
■■ ■■, 2019

<https://doi.org/10.1287/mnsc.2019.3349>

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Abstract. Motivated by bidders' interests in concealing their private information in auctions, we propose an ascending clock auction for unit-demand assignment problems that economizes on bidder information revelation, together with a new general-purpose measure of information revelation. Our auction uses an iterative partial reporting design such that for a given set of prices, not all bidders are required to report their demands, and when they are, they reveal a single preferred item at a time instead of all. Our design can better preserve bidder privacy while maintaining several good properties: sincere bidding is an ex post Nash equilibrium, ending prices are path independent, and efficiency is achieved if the auction starts with the auctioneer's reservation values. Our measurement of information revelation is based on Shannon's entropy and can be used to compare a wide variety of auction and nonauction mechanisms. We propose a hybrid quasi-Monte Carlo procedure for computing this measure. Our numerical simulations show that our auction consistently outperforms a full-reporting benchmark with up to 18% less entropy reduction and scales to problems of over 100,000 variables.

History: Accepted by Chris Forman, Information Systems.

Funding: This research received support from the National Science Foundation of China [Grant 71171052].

Keywords: assignment problem • ascending auctions • privacy preservation • entropy • quasi-Monte Carlo

1. Introduction

Auctions inevitably require bidders to reveal their private demand information. As many have noted, bidders are reluctant to reveal their private valuation (e.g., Ausubel and Milgrom 2002, Sunderam and Parkes 2003) or have “privacy concerns.” One reason is that bidders expect to participate in subsequent activities and negotiations where information contained in their bids can be used against them by competitors and third parties (Ausubel 2004, Rothkopf et al. 1990, Rhodes-Kropf and Katzman 2008, Moldovanu 2012). Also, bidders may not want the public to know their bids—whether they won or lost—for fear of undesirable publicity (Klemperer 2002). Bidders' privacy concerns, if unabated, can have several negative consequences. Bidders may be reluctant to participate in an auction (Sunderam and Parkes 2003) or may hide their true valuations, leading to allocative inefficiency and revenue loss (Ausubel and Milgrom 2002). Therefore, it is important to design auctions in a way that mitigates bidder privacy concerns.

Researchers often argue that dynamic auctions, in which bidders gradually reveal their preferences in multiple iterations, can better preserve bidder privacy (Ausubel 2006, Cramton 2006, Lucking-Reiley 2000,

Bichler et al. 2009). But there has not been much effort to formalize the notion of bidder privacy preservation or to optimize a dynamic auction for preserving bidder privacy. Noting these gaps, we ask the following research questions: (1) Can we design a dynamic auction that better preserves bidder privacy while maintaining desirable economic properties such as efficiency and sincere bidding? (2) Can we formalize bidder privacy preservation so that different auction and nonauction mechanisms can be compared on this dimension? By *bidder privacy preservation*, we are specifically concerned with economizing on the information revelation requirement of an auction design.

Our approach of preserving bidder privacy is complementary to measures aimed to protect “already revealed” bidder private information using encryption and other data security methods. Information protection measures offered by auctioneers (or third-party platforms) are often imperfect. Bidders may not trust the auctioneer's intention or ability to keep their bids secret and not use them against the bidders.¹ Recent data breach incidents from large platforms, including the Internal Revenue Service and certain credit agencies, suggest that the risk of a data

breach is significant and the eradication of such risks far from trivial. In light of this, our approach of economizing on bidder information revelation offers distinct privacy-preservation benefits. Furthermore, our approach can be used jointly with information-protection measures. Auctions that economize on information revelation also make information protection easier.

We explore the privacy-preserving auction design for a class of unit-demand *assignment problems*, which are concerned with assigning m heterogeneous items to n agents, where each agent has a unit demand and private valuations. We choose unit-demand assignment problems as a starting point for privacy-preserving auction designs for a few reasons. Assignment problems with unit demand have well-known applications in matching markets, job assignments, and crew scheduling (Gale and Shapley 1962, Leonard 1983, Crawford and Knoer 1981) and are subproblems in more complex problems such as traveling salesman and vehicle routing (Burkard and Çela 1999). Its recent applications include online display advertising (Feldman et al. 2010), online crowdsourcing marketplaces (Ho and Vaughan 2012), and electric vehicle networks (Clemente et al. 2014). Ideas for solving unit-demand assignment problems have been instrumental for developing solutions for more general network flow and assignment problems (Akgül 1992).

Our research makes the following three main contributions:

- First, we propose an ascending clock auction design for unit-demand assignment problems that offers better protection for bidder privacy than existing benchmarks while maintaining several desirable economic properties (e.g., efficiency and sincerity) and scaling to problems of over 100,000 variables.
- Second, we propose a novel, general-purpose measure of information revelation based on entropy that can be used to compare a wide variety of different auction and nonauction mechanisms.
- Third, we propose a hybrid quasi-Monte Carlo (quasi-MC) method for computing the entropy-based measure of information revelation for ascending clock auctions.

We next discuss the aforementioned contributions in further detail. Our first contribution is an ascending clock auction, which we call a *privacy-preserving ascending clock (PAC) auction*, that economizes on bidder information revelation. In our auction, item prices are posted and may increase when there is excess demand. We iteratively choose subsets of items with excess demands (called *active items*) and ask bidders assigned to these items (or *active bidders*) whether they want to submit a new bid. Based on the new bid, or lack thereof, we determine whether to (1) increase the

prices of the active items simultaneously, (2) reassign the bidder with a new bid, or (3) expand the set of active items. This process continues until no item has excess demand.

Notably, the PAC auction uses an *iterative partial reporting design* in the sense that in each iteration, not all bidders are required to report their demands, and when they are, they only need to reveal a single preferred alternative instead of all. Because of the partial reporting design, we can economize on bidder information revelation. Remarkably, despite partial reporting, the PAC auction maintains several good economic properties, including (1) it stops in a finite number of iterations, (2) sincere bidding at every stage of the auction is an ex post Nash equilibrium, and (3) for a given set of valuations, the auction revenue and ending prices depend only on starting prices (i.e., path independence). Moreover, if a PAC auction starts with the auctioneer's reservation values, then (4) it ends with an efficient allocation and (5) Vickery–Clarke–Groves (VCG) payments for winning bidders.

In contrast, most existing ascending auctions for unit-demand assignment problems that demonstrate similar economic properties assume *full reporting*, in the sense that *all* bidders must report, at any iteration, their entire *demand set*—that is, all preferred items at the current prices.² These auctions use fully reported demand sets to find an *overdemanded set*—a set of items with the property that the number of bidders who *only* prefer items in the set exceeds the number of items in the set. Full reporting allows the auctioneer to immediately determine an appropriate set of items for price increase but has two main drawbacks. First, as noted by Demange et al. (1986), Gul and Stacchetti (2000), and Perry and Reny (2005), it is “excessive” in the sense that bidders must report, at every price, their entire demand sets. An accidental omission of one item from reported demand sets can cause an inconsistency with later reports. Second, as we will show, full reporting is unnecessary for achieving the above-mentioned economic properties. Requiring full reporting can exacerbate bidder privacy concerns.

Our second contribution is a novel, general-purpose measure of information revelation. This measure captures the amount of information gained about a bidder by calculating the difference between the entropies of the posterior and the prior distributions of the bidder's valuation. We use an entropy-based measure because it satisfies a long list of desirable properties (see Section 5.1 for details). Moreover, the measure is general enough to be used by any number of auction and nonauction mechanisms. To our knowledge, this is the first time entropy reduction is used for measuring bidder privacy preservation in an auction or a mechanism design setting.

Our third contribution is a practical approach for computing the information-revelation measure for ascending auctions. In an ascending auction with many bidders and items, calculating the posterior entropy amounts to a high-dimensional integration problem. To solve this problem, we first identify and record informative events in an ascending auction as a set of constraints on bidder valuations and then propose an algorithm for computing high-dimensional entropy reduction using a hybrid quasi-MC approach. Our numerical experiments demonstrate the viability of our computational approach.

2. Related Literature

Our research is related to two literature streams: the dynamic auction literature and the literature on information revelation and entropy measure of information.

2.1. Dynamic Auctions

The literature of dynamic auctions for the assignment problem and its generalization starts with the seminal work of Demange et al. (1986) (henceforth DGS). DGS propose a dynamic auction that requires full reporting—that is, each bidder reports his or her entire demand set in each iteration. The auctioneer computes a *minimal overdemanded set*, which is an overdemanded set with the property that none of its proper subsets is an overdemanded set. The auctioneer raises the price of the minimal overdemanded set by one unit and then asks all bidders for their demand sets again. This process continues until there is no overdemanded set, which terminates the auction. DGS show that in their auction, which always starts with sellers' reservation values, sincere bidding is an equilibrium, the auction ends with a unique minimum equilibrium price vector, and the auction is efficient.

The same full-reporting requirement, together with the computation of minimal overdemanded sets, is used in the auctions of Gul and Stacchetti (2000), Ausubel (2006), and de Vries et al. (2007), which extend beyond unit demands. Sankaran (1994) points out that the computation of minimal overdemanded sets is expensive and shows that the auction ends with the same price vector if minimal overdemanded sets are replaced by sets produced by the Ford–Fulkerson algorithm (Ford and Fulkerson 1962). This idea is further generalized by Andersson et al. (2013), who find *special sets in excess demand*, which require each subset T of a set S in excess demand to have more bidders than items, and these bidders only prefer items in S . In the implementations of the DGS auction by Sankaran (1994) and Andersson et al. (2013), however, the same full-reporting requirement is used.

de Vries et al. (2007) show that the DGS auction can be interpreted as an application of the primal–dual

algorithm for the assignment problem. The primal–dual algorithm discussed in (de Vries et al. 2007) still requires bidders to submit their entire demand set (i.e., it requires full reporting).³

Interestingly, DGS note that their auction is “difficult to implement in realistic situations” and propose an “approximate” auction to reduce the reporting requirement. In the approximate auction, each item can accept at most one bidder, and a new bidder displaces the incumbent bidder and increases the price by a small increment. DGS show that, with a small enough increment, and an assumption of sincere bidding, this auction can be arbitrarily close to the efficient DGS auction. However, the approximate DGS auction has two drawbacks. First, it can lead to unnecessary reassignments. Consider two bidders who both value an item at 100. Starting from a price of 0, the two bidders must take turns to be the incumbent, and it takes 100 reassignments for the auction to end. Second, the DGS approximate auction provides no guarantee for sincere bidding. Without sincere bidding, it is unclear where the auction would end or whether it would still approximate the DGS auction. Our auction does not have the aforementioned drawbacks: our auction replicates the economic properties of the DGS auction with a partial reporting design.

Our research, in a broad sense, is related to recent research on multi-item dynamic auctions that considers more general demand functions (Gul and Stacchetti 2000, Bikhchandani and Ostroy 2002, Ausubel and Milgrom 2002, Ausubel 2006, de Vries et al. 2007, Perry and Reny 2005, Mishra and Parkes 2007, 2009), though these papers do not pay much attention to the bidder information revelation. There has been some work in the combinatorial auctions literature on how to reduce bidder reporting (Parkes 2001, 2002), but this stream of research focuses more on the winner computation and communication complexities associated with reporting rather than on preserving bidder privacy.

Our research is also broadly related to the literature on practical dynamic auction designs. Information system researchers have examined a few practical design issues in iterative combinatorial auctions, such as pricing rules (Bichler et al., 2013, 2009, 2017) and bidder decision support (Adomavicius and Gupta 2005, Adomavicius et al. 2013, Petrakis et al. 2013). There is also a stream of research on dynamic auction designs for smart markets (Bapna et al. 2008, 2011, Bichler et al. 2010). Our research adds to these literatures with a new dimension of bidder privacy preservation.

2.2. Bidder Information Revelation

Before the start of an auction, the auctioneer has a prior regarding the valuations of the bidders. After the

auction terminates, the auctioneer can update this prior and obtain a posterior based on the bids he or she has observed. Our measure of information revelation is simply a measure of entropy reduction that results from such updating. The concept of entropy reduction is introduced by Lindley (1956) and has since been used widely by statisticians and information theorists. Lindley's idea for measuring the information revealed by a statistical experiment (e.g., an event) is to simply compute the difference between the uncertainty contained in the posterior and the uncertainty contained in the prior, where the uncertainty contained in a random variable is measured using Shannon's entropy. The seminal work of Claude Shannon on entropy in the late 1940s has made it clear that information is essentially a statistical concept, and since then, there has been a growing body of literature on using entropy to quantify information gain/loss in business and economic settings starting with the work of Marschak (1959) and Arrow (1971) and continuing over the years to more recent work by Marschak and Radner (1972), Demski (1973), Sims (2003), Peng (2005), and Cabrales et al. (2013).

Our measure of information revelation is very different from the notion of informational surplus (Parkes 1999), which measures the ratio between a bidder's final bid on items and his or her true valuations. The informational surplus has more to do with bidders' surplus than the amount of information revealed. An example can illustrate this point. Consider an ascending auction with a buyer who has a private valuation of $v = 50$. Suppose initially that the seller believes that the valuation of the buyer is uniformly distributed over $[50, 100]$. Suppose that the last bid of the buyer is 50. The informational surplus for this bidder is 1, indicating that the bidder has surrendered his or her entire surplus. However, the prior and the posterior of the seller are the same, and the information revealed is 0 by our measure. We note also that the computation of information surplus requires knowledge of a bidder's private information and thus cannot be carried out by the auctioneer.

Bidder privacy is also addressed in the literature of secure multiparty computation (SMC), which follows an entirely different approach. SMC, when applied to auctions, allows several computing agents (bidders or third parties) to jointly compute the auction outcome over inputs while keeping those inputs private. As reported by Brandt and Sandholm (2008), most SMC protocols rely on at least one of the following three conditions: (1) a certain fraction of the computing agents is trusted not to reveal any private information, (2) malicious parties are limited to polynomially bounded computational power, and (3) the existence of one-way functions.⁴ There is an increasing interest in deploying SMC protocols to the real world.

However, the implementation of SMCs has been very limited because of the challenges in finding a feasible SMC protocol, the high computational cost, and a general lack of understanding of the technology among the public (Bogetoft et al. 2009, Orlandi 2011). Though some prototype SMC protocols exist for auctions, they are mostly for specific types of one-shot, sealed-bid auctions, and they tend to solve relatively simple multiparty computation problems, often at a high computational cost (Brandt 2006, Brandt and Sandholm 2005, Baudron and Stern 2002, Harkavy et al. 1998, Juels and Szydlo 2003, Montenegro and Lopez 2014). Conceptually, our approach for bidder privacy, which allows bidders to reveal less information, is complementary to the SMC approach: auctions with lower information revelation (such as ours) may allow for computationally less complex SMC designs.

The problem of designing an auction to preserve bidder privacy is also different from the disclosure problems in auctions, where the main concern is whether the auctioneer or the bidders have incentives to disclose their private information. Examples of the latter topic include whether the auctioneer (or seller) should disclose his or her private information about the item (Milgrom and Weber 1982, Lewis 2011) and whether bidders have incentives to disclose their valuation information (Board 2009, Tan 2016).

3. Auction Design

Consider an assignment problem with m items and n bidders (or buyers). We focus on cases where each bidder has a unit demand: that is, although the bidder may have positive valuations for many items, he or she is only interested in buying one of them.

We denote $\mathcal{J} = \{0, 1, 2, \dots, m\}$ as a set of alternatives that includes m items and an outside option (indexed by 0). For $i \in \{1, 2, \dots, n\}$ and $j \in \mathcal{J}$, denote $v_{ij} \in [0, \infty)$ as bidder i 's private valuation for alternative j . For notational convenience, we use v_{0j} to denote the auctioneer's (indexed by 0) reservation value for alternative j . We fix v_{00} , the auctioneer's reservation value for the outside option, at 0. See Table 1 for a summary of notation.

The proposed PAC auction is a special case of ascending clock auctions. In a typical clock auction (Ausubel and Cramton 2006), the auctioneer announces prices, one for each of the items being sold, and the bidders respond with the item(s) they prefer at the current prices—that is, *their bid(s)*. Prices for items with excess demand increase until bidders submit different bids. This process is iterated until there is no excess demand. Our auction follows the general pattern but has special rules for bidder activities and price adjustments.

In the PAC auction, each iteration t (except for the last one) consists of a *reassignment phase* followed by

Table 1. Notation

Notation	Interpretation
m	Number of items
n	Number of bidders
\mathcal{G}	A set of alternatives $\{0, 1, 2, \dots, m\}$ including m items and the outside option 0
v_{ij}	bidder i (or the auctioneer, if $i = 0$)'s valuation for alternative j
\mathbf{v}_i	$\mathbf{v}_i = (v_{i0}, v_{i1}, \dots, v_{im})$
t	Index the iterations of an auction
p_j^t	the price of item j at the beginning of iteration t
\mathbf{p}^t	$\mathbf{p}^t = (p_0^t, p_1^t, \dots, p_m^t)$
x_i^t	the item assigned to agent i at the end of iteration t
\mathbf{x}^t	$\mathbf{x}^t = (x_1^t, \dots, x_n^t)$
$D_i(\mathbf{p})$	The demand set for a bidder i given the price vector \mathbf{p} .
$RD_i(\mathbf{p})$	The revealed demand set for a bidder i given the price vector \mathbf{p} .
Ω	The set of auction outcomes $\{(x, \mathbf{p})\}$ where every bidder is sincere.
Ω_{-i}	The set of auction outcomes $\{(x, \mathbf{p})\}$ where every bidder but i is sincere.

a *price-increase phase*. In a reassignment phase, prices of all items remain fixed. The auctioneer chooses a subset of items with excess demand, called an *active set* \mathcal{A} , and iteratively refines it by reassigning *marginal bidders* (i.e., an active bidder who is indifferent between his or her assigned item and an alternative outside the active set) or expanding the set until one obtains an active set that has no marginal bidder—which indicates an overdemanded set. In the price-increase phase, the auctioneer increases the prices of the active set while bidder assignments remain fixed, until a marginal bidder appears.

Denote $\mathbf{p}^t = (p_0^t, p_1^t, p_2^t, \dots, p_m^t)$ as a price vector at the beginning of iteration t . The price for the outside option p_0^t is fixed at 0 at all times. We often omit the superscript t for simplicity.

We define bidder i 's *demand set* for a price vector \mathbf{p} as the set of alternatives that maximize i 's utility:

$$D_i(\mathbf{p}) = \arg \max_{j \in \mathcal{G}} (v_{ij} - p_j), \forall i = 1, 2, \dots, n. \quad (1)$$

We further define a bidder i 's *revealed demand* for a given price vector \mathbf{p} , $RD_i(\mathbf{p})$, as a set of preferred alternatives for this price vector, as inferred from his or her current and past bids. For example, if a bidder's current bids are $\{1, 3\}$ (i.e., he or she is indifferent between items 1 and 3), his or her revealed demand should include $\{1, 3\}$. If we further know that the bidder's bid was item 4 when $p_1 = 20$ and $p_4 = 25$ and the current prices are $p_1 = 21$ and $p_4 = 26$, then we infer that the bidder still prefers item 4. Thus, the bidder's revealed demand includes $\{1, 3, 4\}$.

The PAC auction maintains a provisional assignment, which is a mapping from bidders to items. We denote such an assignment as $\mathbf{x} = (x_1, x_2, \dots, x_n)$, where $x_i \in \{0, 1, \dots, m\}$ is the alternative (an item or

the outside option) assigned to bidder i . For example, $x_1 = 2$ represents that item 2 is assigned to bidder 1. If a bidder is assigned to the outside option, we say the bidder is *unassigned*. Similarly, an item is *unassigned* when no bidder is assigned to it. We say a bidder and an item are *singular* if the bidder is solely assigned to the item.

We now describe the PAC auction in detail (see Figure 1 for a flowchart). To facilitate the description, we imagine that for each item, there is a clock displaying the current price of the item for all bidders to see. We say a clock is *active* if the associated item is part of the active set.

1. (Initialization) The iteration counter t is set to 0. All clocks are initially inactive, and the prices are set to \mathbf{p}^0 . Each bidder names a single alternative (an item or the outside) as his or her initial bid and is provisionally assigned to it.

2. (Reassignment) While the clocks are inactive, the auction proceeds as follows:

(a) (Reassign the marginal bidder) If there is a marginal bidder i , and i is not singular (i.e., not solely assigned to an item), then reassign i to his or her new bid. If i is singularly assigned, say to item 1, find a replacement bidder i' among active bidders such that i' 's revealed demand includes item 1. Reassign the replacement bidder i' to 1 before assigning i to his or her new bid so that item 1 maintains at least one bidder.⁵

(b) (Termination) If no item has more than one bid, the auction ends. Each assigned bidder receives his or her assigned item and pays the current price of the item. The unassigned bidders do not pay.

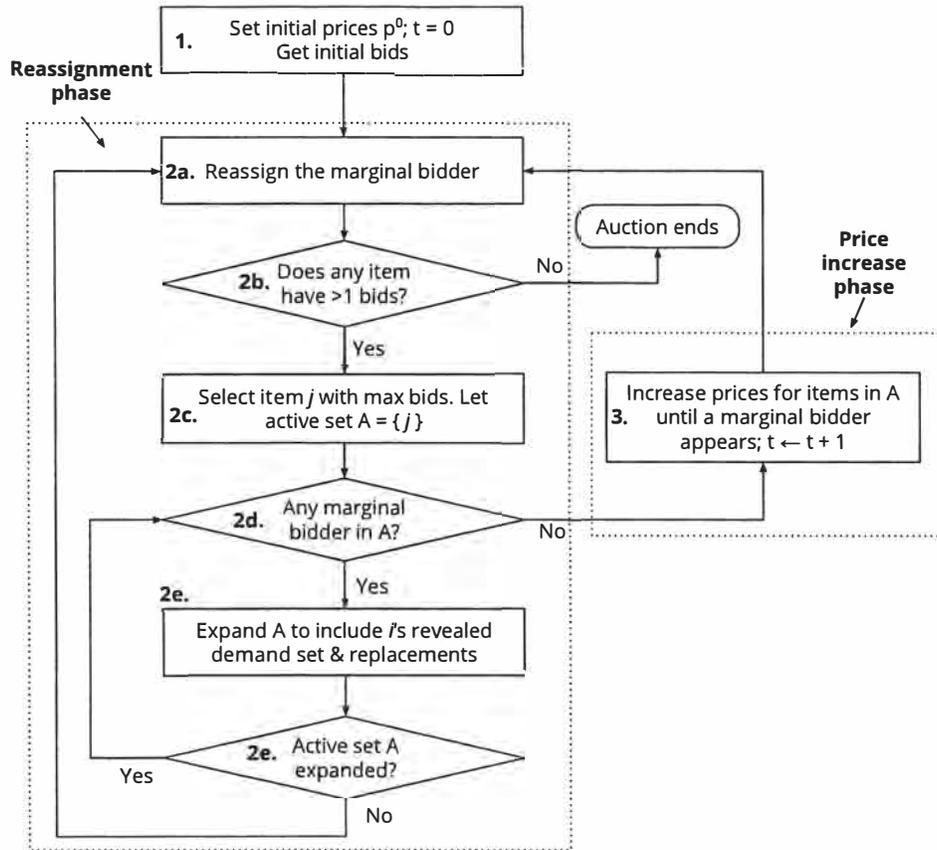
(c) (Choose an initial active set) Among items with more than one bid, choose the one, say j , with the largest number of bids as the new active set $\mathcal{A} = \{j\}$.⁶ Break ties arbitrarily.

(d) (Bid inquiry) Active bidders are asked whether they are marginal. If there are multiple marginal bidders, break ties arbitrarily. If there is none, the reassignment phase ends, and the auction enters a price-increase phase (step 3). Otherwise, proceed to the next step.

(e) (Active set expansion) Expand the active set \mathcal{A} to include the marginal bidder's revealed demand set. This prevents the marginal bidder from switching back and forth between his or her reportedly indifferent items and reduces unnecessary bid revisions. Furthermore, if the marginal bidder is singular, expand the active set to include his or her replacement.⁷ If this step results in an expanded active set, then go to step 2d to query a new set of active bidders. Otherwise, reassign the marginal bidder (step 2a).

3. (Price increase) Activate all the clocks in the active set. The prices of active clocks increase synchronously, and the active bidders are asked for new bids as prices increase, until a marginal bidder appears.⁸

Figure 1. Flowchart of the PAC Auction Design



As soon as the marginal bidder appears, the price-increase phase ends. Again, if there are multiple marginal bidders, break ties arbitrarily. Deactivate the clocks, increase the iteration counter $t \leftarrow t + 1$, and go to step 2a to reassign the marginal bidder.

For example, consider a PAC auction with five bidders and three items with valuations given by the payoff matrix y in the first iteration of Table 2. For the purpose of this illustration, we assume the valuations for the outside option and the auctioneer's reservation values are 0. We also assume that all bidders bid sincerely—that is, they bid on items that provide them with the highest payoff at current prices.

As seen from Table 2, in iteration 1, the auctioneer first sets the starting prices to 0 and bidders bid on their most preferred alternatives. Item 2 has excess demand and is chosen as the initial active set. At current prices, bidder 3 is marginal with a new bid on item 3. So the auctioneer reassigns bidder 3 to item 3 and chooses a new active set to be item 2, which has two bids. With the new active set $\{2\}$, neither active bidder is marginal, so the auctioneer activates the clock 2 and increases the price by 5, when a marginal bidder 5 appears. In iteration 2, the marginal bidder 5 is reassigned to his new bid, item 3. The auctioneer chooses item 3 to be the new active set. The marginal

bidder 3's revealed demand set is $\{2, 3\}$, so the auctioneer expands the active set to include item 2. With the new active set $\{2, 3\}$, there is no marginal bidder, so the auctioneer activates clocks $\{2, 3\}$ and increases the prices by 5, when a new marginal bidder 2 appears. In iteration 3, the marginal bidder 2 is reassigned to his or her new bid of item 1. The auctioneer chooses item 3 to be the new active set. The marginal bidder 5 has a revealed demand set of $\{2, 3\}$, so the auctioneer adds 2 to the active set. With the new active set $\{2, 3\}$, the new marginal bidder is 4, and he or she is singular. His or her replacement bidder 5 is active and nonsingular. So the auctioneer reassigns bidder 5 to item 2 and then assigns marginal bidder 4 to his or her new bid of item 1. The auctioneer chooses the new active set to be item 1, and the marginal bidder 4 has a revealed demand of $\{1, 2\}$; therefore, the auctioneer adds 2 to the active set. With the active set $\{1, 2\}$, the marginal bidder 2 has a revealed demand of $\{1, 3\}$, so the auctioneer adds 3 to the active set. With the active set $\{1, 2, 3\}$, there is no marginal bidder, so the auctioneer activates clocks $\{1, 2, 3\}$ and increases the prices by 19, when a new marginal bidder 1 appears. In iteration 4, the marginal bidder 1 drops out. The auctioneer chooses the new active set to be item 2, the marginal bidder 2 has a revealed

Table 2. An Example of the PAC Auction

t	Prices p & Payoffs y			Bidders' current bid					Num of bids at item			A	MB	to	Remark		
	1	2	3	#	1	2	3	4	5	1	2					3	
1	p	0	0	0	#1	1	3	②	2	2	1	③	1	2	3	3	bidder 3: 2→3
	y	19	3	18	#2	1	3	3	2	2	1	②	2	2	-	-	p_2+5
		21	24	26	2 inquiries and 1 bid revision												
		0	25	25													
		22	32	27													
	17	30	25														
2	p	0	5	0	#1	1	3	3	2	②	1	②	2	2	5	3	bidder 5: 2→3
	y	19	-2	18	#2	1	3	3	2	③	1	1	③	3	5	-	$\mathcal{A} \cup \{2\}$
		21	19	26	#3	1	3	3	2	3	1	①	③	3,2	-	-	$\{p_2, p_3\} + 5$
		0	20	25	2 inquiries and 1 bid revision												
		22	27	27													
	17	25	25														
3	p	0	10	5	#1	1	③	3	2	3	1	①	③	3,2	2	1	bidder 2: 3→1
	y	19	-7	13	#2	1	1	3	2	③	2	1	②	3	5	-	$\mathcal{A} \cup \{2\}$
		21	14	21	#3	1	1	3	②	3	2	①	②	3,2	4	-	MB 4 is singular
		0	15	20	#4	1	1	3	2	③	2	①	②	3,2	5	2	bidder 5: 3→2*
		22	22	22	#5	1	1	3	②	2	2	②	①	3,2	4	1	bidder 4: 2→1
		17	20	20	#6	1	1	3	①	2	③	1	1	1	4	-	$\mathcal{A} \cup \{2\}$
					#7	1	①	3	1	2	③	①	1	1,2	2	-	$\mathcal{A} \cup \{3\}$
					#8	1	1	3	1	2	③	①	①	1,2,3	-	-	$\{p_1, p_2, p_3\} + 19$
			5 inquiries and 3 bid revisions														
4	p	19	29	24	#1	1	1	3	1	2	③	①	①	1,2,3	1	0	bidder 1: 1→0
	y	0	-26	-6	#2	0	①	3	1	2	②	1	1	1	2	-	$\mathcal{A} \cup \{3\}$
		2	-5	2	#3	0	1	3	①	2	②	1	①	1,3	4	-	$\mathcal{A} \cup \{2\}$
		-19	-4	1	#4	0	1	3	1	2	②	①	①	1,3,2	-	-	$\{p_1, p_2, p_3\} + 1$
		3	3	3	3 inquiries 1 bid revisions												
	-2	1	1														
5	p	20	30	25	#1	0	1	3	1	②	②	①	①	1,3,2	5	-	MB 5 is singular
	y	-1	-27	-7	#2	0	1	3	①	2	①	②	①	1,3,2	4	2	bidder 4: 1→2*
		1	-6	1	#3	0	1	3	2	②	①	①	①	1,3,2	5	0	bidder 5: 2→0
		-20	-5	0	#4	0	1	3	2	0	1	1	1	-	-	-	the end
		2	2	2	0 inquiry and 2 bid revisions												
	-3	0	0														

9 : Active bidder ② : Marginal bidder ③ : Active item
 A : Active set MB : Marginal bidder to : Marginal bidder's new bid
 * : a replacement step

demand set of {1,3}, so the auctioneer adds 3 to the active set. With the active set {1,3}, an active bidder 4 has a revealed demand of {1,2}, so the auctioneer adds 2 to the active set. With the active set {1,2,3}, there is no marginal bidder, so the auctioneer activates clocks {1,2,3} and increases the prices by 1, when a new marginal bidder 1 appears. In iteration 5, the marginal bidder 5 is singular at item 2. His or her replacement bidder 4 is active and nonsingular, so the auctioneer reassigns bidder 4 from item 1 to item 2. The auctioneer then lets the marginal bidder 5 drop

out. At this point, neither item has excess demand, so the auction ends.

From the description of the auction procedure and the preceding example, we make the following observations about the PAC auction:

Observation 1. An active bidder cannot bid on the same item twice during a reassignment phase.

Observation 2. Bidders who drop out can never reenter.

Observation 3. Unassigned items must be at their starting prices.

Observation 1 states that an active bidder cannot make a round trip within a single reassignment phase. This is because once the bidder becomes marginal, we expand the active set to include all items in his or her revealed demand set, so he or she can no longer bid on them (refer to step 2e). There is no reentry into the auction (Observation 2) because to leave the outside option, the outside option needs to be active. However, the outside option, by design, can never be part of the active set.⁹

To see Observation 3, we first argue that once an item gets its first bidder, it will maintain at least one bidder in subsequent bidding. By our auction rule, only an active item may lose a bidder. If an active item has two or more bidders, the argument holds trivially. If an active item has only one bidder, which can happen only when it becomes active via active set expansion, our auction procedure as described in step 2a ensures that it maintains at least one bidder. Thus, an unassigned item must have no prior bids. Because an item with no prior bids has never been active and the auctioneer only increases prices of active items, an unassigned item must be at its starting price.

Importantly, the PAC auction implements a partial reporting design that can reduce information revelation. The auctioneer only needs to know the new bid of one marginal bidder if there are multiple bidders. For example, in step 3 of iteration 5, both bidders 3 and 5 are marginal, but because 5 drops out, 3 never needs to reveal that he or she is indifferent between staying and dropping out. Moreover, the marginal bidder only needs to name a single alternative from his or her demand set, even though his or her demand set may be larger. For example, in step 5 of iteration 3, marginal bidder 4 is indifferent between three items, but he or she does not reveal that he or she is indifferent between items 2 and 3 (he or she never will). As can be seen from these examples, partial reporting can lead to less information revelation. This is especially true when there are *ties* in the auction process, whether between competing marginal bidders or between several preferred alternatives of the same marginal bidder.

It is interesting to compare the PAC auction with the Hungarian method for assignment problems (Kuhn 1955). As the first primal–dual approach, the Hungarian method alternates between improving the dual variables (u, v), which may be interpreted as *surplus* for bidders and *prices* for items, respectively, and primal variables x , which is a feasible solution for a restricted primal problem. The Hungarian method starts from an empty assignment and augments it by assigning additional bidder–item pairs until it assigns all bidders/items. Our PAC auction alternates between adjusting the assignment (primal) and item prices (dual), but it is a dual algorithm rather than a

primal–dual one. In particular, our auction maintains a provisional assignment for all agents, which is an *infeasible* primal vector, and pivots it until it becomes feasible. Furthermore, the Hungarian method and algorithms inspired by it typically assume complete information or full reporting of demands,¹⁰ whereas the PAC auction operates on partial information about demand sets.

4. Auction Properties

In this section, we analyze our auction’s equilibrium bidding behaviors, efficiency, revenue, and ending prices. We also study the relationship between our auction and the VCG mechanism.

In dynamic auctions, the final outcomes depend on the auction process, which may not be unique because of tie breaking by the auctioneer or bidders. Furthermore, when bidders are not sincere, we do not know what the end prices or assignments will be; in fact, we do not know if the auction will end. Hence, to limit the number of possible outcomes, we start by establishing that sincere bidding is an *ex post* Nash equilibrium.¹¹

4.1. Sincere Bidding

Sincere bidding in our auction context is defined as follows.

Definition 1 (Sincere Bidding). An active bidder bids *sincerely* if his or her actions are consistent with his or her demand set; that is, the bidder submits a new bid if and only if his or her demand set includes an alternative that is inactive, and the new bid is indeed from the bidder’s demand set. A bidder is sincere if he or she always bids sincerely.

The definition implies that in the initial and subsequent bids, a sincere bidder can only bid on items in his or her demand set, and he or she must submit a new bid if his or her demand set includes an alternative that is inactive.

To study whether a bidder has incentive to deviate from sincere bidding, we compare the auction outcomes when bidders are sincere and when all but one bidder is sincere. But first, we need to know whether the auction actually ends. Clearly, if there are two or more insincere bidders, the auction can continue indefinitely (though this may not be in their best interests). Fortunately, when there is at most one insincere bidder, we have the following.

Lemma 1. *If there is at most one insincere bidder, the auction must end in finite iterations.*

Proof. All proofs are in the appendix. \square

We additionally note there can be only a finite number of bid revisions within an iteration. This is because our active set expansion mechanism prevents

the marginal bidder from bidding on items toward which he or she has already been revealed to be indifferent. The actual bid revisions per iteration are much fewer than what is theoretically possible, which we later show through our numerical experiments.

We now consider auction outcomes with and without an insincere bidder. For a given set of valuations and starting prices, let Ω denote the set of all possible outcomes when all bidders are sincere, and let Ω_{-1} denote those when all but bidder 1 are sincere.¹² We call outcomes in Ω *sincere outcomes* and outcomes in Ω_{-1} *bidder 1 insincere outcomes*. Because the insincere bidder can also bid sincerely, $\Omega \subseteq \Omega_{-1}$.

For a given assignment x , we further denote $x(I)$ as the assignment for a subset of bidders I . For example, $x(1, 2, 3) = (3, 2, 5)$ means that bidders 1, 2, and 3 are assigned to items 3, 2, and 5, respectively.

Lemma 2. Consider a sincere outcome $(x, p) \in \Omega$ and a bidder 1 insincere outcome $(x', p') \in \Omega_{-1}$. If bidder 1 strictly prefers (x', p') to (x, p) , there must be a set of bidders I such that $1 \in I$, $x(I)$ is a permutation of $x'(I)$, and

$$p'_j < p_j, \forall j \in x(I). \quad (2)$$

Lemma 2 says that for an insincere bidder to profit from manipulating the auction's outcome, he or she must be part of a group that collectively obtains the same items (possibly with permutation) at strictly lower prices than the sincere auction outcome. Such a group of bidders cannot exist, as per the next lemma.

Lemma 3. Consider a sincere auction outcome $(x, p) \in \Omega$ and a bidder 1 insincere outcome $(x', p') \in \Omega_{-1}$, where bidder 1 weakly prefers (x, p) to (x', p') .

The group described in the Lemma 2 cannot exist because there is a *last bidder* problem: to depress prices for the items assigned to the group, there must be an out-group bidder who prematurely exits the competition for these items, which could not happen because all out-group bidders are sincere.

With Lemma 3, we immediately have the following proposition.

Proposition 1. Bidding sincerely is an *ex post* Nash equilibrium.

We note that our proof of sincere bidding does not depend on the actual auction path. This implies that the auction's sincereness is robust with respect to tie-breaking rules. More important, we establish Proposition 1 without reference to the efficiency of auction outcomes. Our proof of Proposition 1 (including the lemmas leading to it) is based on simple combinatorial arguments that do not require establishing the existence of minimum Walrasian prices or the equivalence between minimum Walrasian prices and VCG payments.

4.2. Efficiency and Connection to VCG

Starting the auction with arbitrary prices might lead to an inefficient outcome. However, our auction can achieve efficient outcomes for a specific set of starting prices.

Proposition 2. When the starting prices are the auctioneer's reservation valuations, the auction outcome is efficient.

Intuitively, our auction achieves efficiency if both the auctioneer and the bidders are *sincere*. Proposition 1 guarantees that bidders are sincere. When the starting prices are equal to the auctioneer's reservation values, the auctioneer is also *sincere*, and the auction is efficient.

The next proposition shows that when starting prices are equal to the auctioneer's reservation valuations, the auction implements the VCG mechanism. Therefore, a special case of our auction can be viewed as a decentralized implementation of the VCG mechanism.

Proposition 3. If an auction starts with the auctioneer's reservation values, the bidders' final payments coincide with their VCG payments for the same final assignment.

We note that the VCG assignment may not be unique. In our auction, because of tie breaking by bidders and the auctioneer, the auction may end with different sets of winners and/or the same bidder winning different items. Proposition 3 only ensures that the payments will be the same *if our auction and the VCG mechanism result in the same assignment*.

4.3. Revenue and Ending Prices

Auction revenue and ending prices are also important considerations in auction design. An important decision for the auctioneer is to set the starting prices, which is analogous to setting reserve prices in one-shot auctions. Given the importance of choosing reserve prices in the optimal auction design literature (Myerson 1981), a revenue-maximizing auctioneer may want to set starting prices above his or her true reservation values, especially if he or she has some knowledge of the distribution of bidders' valuations.

From the revenue perspective, an important question is whether our auction can end with different ending prices for a given starting price vector, knowing that our auction can end with different final assignments. The following result establishes that the final prices are the same.

Lemma 4. If $(x, p) \in \Omega$ is an auction outcome, then p is the smallest price vector among all possible auction outcomes Ω .

Lemma 4 implies that our auction must end with a unique set of prices (because there can only be one smallest price vector for Ω). When the starting prices are 0, the auction terminates at a price vector that is minimal in the lattice of equilibrium prices, which is consistent with the ascending auction literature

(Demange et al. 1986, Ausubel 2006, de Vries et al. 2007, Mishra and Talman 2010, Andersson et al. 2013).¹³ Even though the auction may end with different assignments, the final price vector is always the same (path independence). Therefore, the auction revenue is also unique and path independent. The next result follows immediately from Lemma 4.

Proposition 4. *Auction revenue and ending prices are path independent. Moreover, identical items with the same starting price must have the same ending price.*

5. Bidder Information Revelation

Our first step toward studying the bidder privacy preservation is to formalize the notion of bidder information revelation. We start with a general discussion of how to measure the reduction in the uncertainty surrounding some random variable (i.e., the state of the world) as a result of observing some event. We then specialize this discussion to the contexts of auctions.

5.1. Measuring Information Revealed by an Event

One intuitive way of measuring the information revealed by a particular event is to compare our prior distribution regarding the random state of the world to the posterior conditional on the occurrence of this event. It is natural to require the measure of information revelation to satisfy the following list of desirable properties:

- If the posterior is identical to the prior, then the event has not revealed anything, and the measure should be 0.
- If the posterior has smaller support than the prior, the event is informative, and the measure should be positive. The most informative event will yield a posterior that is concentrated on a single point; the measure of information revealed should be maximal in this case.
- If the conditional probabilities of the state of the world given event E_1 and those given event E_2 are the same, then E_1 and E_2 are equally informative, so their measures should be the same.
- The measure of information revealed by simultaneously observing two independent events is the sum of the information revealed by the two events and should be the same if we observe the two events sequentially.
- If we already know the state of the world the measure of information revealed by any event must be 0.

Among the available information revelation measures, the measure of entropy reduction, first introduced by Lindley (1956), is most natural because it satisfies a long list of desirable properties that include all the aforementioned ones (Shannon and

Weaver 1964, Basu 1975) and has been widely used by statisticians and information theorists.

Lindley's entropy reduction is built on Shannon's information entropy, which is the standard measure used in information theory for quantifying the average uncertainty contained in a random variable.¹⁴ In general, the *entropy* of a random variable X that takes the values $\{x_1, x_2, \dots, x_k\}$ and has a probability mass function $P(x)$ is given by

$$\text{(Entropy)} \quad H(X) = - \sum_{k=1}^K P(x_k) \log_2 P(x_k), \quad (3)$$

with the convention that $0 \cdot \log_2 0 = 0$. Note that the entropy of X depends only on its probability mass function, not on the actual values that X can take.

Intuitively, the entropy of X is the average number of bits needed to describe this random variable. For example, if X is the result of a coin flip, the entropy of X would be $H(X) = -2 \times 0.5 \times \log_2(0.5) = 1$ bit. Alternatively, if someone else is trying to find out what the realized outcome of a random variable X is by asking yes/no questions, entropy is the expected length of the shortest sequence of binary questions needed to determine the outcome. Consider, for example, a random variable X with four possible outcomes a, b, c , and d . Suppose that $P(a) = 1/2$, $P(b) = 1/4$, and $P(c) = P(d) = 1/8$. The shortest sequence of binary questions that minimizes the expected number of questions asked should start with the question, "Is the outcome a ?" If the answer is no, then the next question should be, "Is the outcome b ?" If the answer is still no, then the final question should be, "Is the outcome c ?" The resulting expected number of questions needed to determine the outcome is $7/4$, which is equal to the entropy of X obtained using Shannon's equation (3).

We measure the information revealed by an event E about a random variable X as the reduction in entropy of X resulting from observing the event E . More specifically, it is

$$\text{(Entropy Reduction)} \quad R(X; E) = H(X) - H(X|E), \quad (4)$$

where $H(X)$ is again the entropy of X , and $H(X|E)$ is the entropy of X conditional on the occurrence of E .¹⁵ The formulas defining $H(X)$ and $R(X; E)$ can be easily generalized to the case of continuous random variables.

Returning to our last example, suppose that the event E is "the outcome is neither a nor b ." Then, conditional on this event, $X|E$ is a new random variable with equally likely outcomes c and d . The outcome of $X|E$ can be determined with a single binary question, and hence $H(X|E) = 1$, and the measure of information revealed by the event E is $H(X) - H(X|E) = 7/4 - 1 = 3/4$. By contrast, if the event E' is "the outcome

is a ," then the information revealed by E' is maximal, and the measure of information revealed is $H(X) - H(X|E') = 7/4 - 0 = 7/4$.

Finally, our measure of information revelation allows us to treat information as a homogeneous good (measured in bits), which, in turn, allows us to compare information revelation across auction and nonauction mechanisms. Note that we are measuring the amount of the revealed information and not its value; the former does not depend on who receives the information, whereas the latter does.

5.2. Capturing Informative Events in Ascending Auctions

The amount of information revealed in an ascending auction is generally path dependent. For example, bidding on item 1 throughout the auction reveals different information than first bidding on item 2 and then item 1. In general, at any moment in an auction, given bidder i 's revealed demand RD_i and the price vector \mathbf{p} , we learn that

$$v_{ij} - p_j \geq v_{il} - p_l, \forall l \in \mathcal{J}, \forall j \in RD_i. \quad (5)$$

Thus, events in an auction effectively impose a set of constraints, in the form of Equation (5), on a bidder's valuations. In the event that a bidder announces his or her marginal status but is not picked to submit a new bid, his or her revealed demand set is uncertain. For simplicity, in what follows, we ignore the information revealed through a mere announcement of marginal status. Alternatively, one can assume that the auction is designed in such a way that any announcement of marginal status would block other attempts to announce one's marginal status.

Not all events are informative about a bidder. During a price-increase phase, a bidder would still prefer his or her assigned item, if the price of the assigned item does not increase. So we learn nothing new about the inactive bidders. We learn something new about the active bidders as prices increase, and we learn the most at the end of the price-increase phase, when the prices are the highest. During a reassignment phase, the prices remain constant, and we learn something new about a bidder only when his or her revealed demand set has changed, which occurs when an active bidder submits a new bid. Therefore, it is sufficient to capture the following events about a bidder: *when the bidder submits an initial bid, submits a new bid, or is an active bidder at the end of a price-increase phase.*

Because information about a bidder can be represented by the bidder's revealed demands at various informative events, we can record a bidder i 's *information set* C_i as a set of demand-price pairs $\{(j, \mathbf{p})\}$, where each pair (j, \mathbf{p}) represents the bidder demanding

alternative j at price \mathbf{p} . For example, in the auction example given in Table 2, bidder 3's information set is

$$C_3 = \{(2, (0, 0, 0)), (3, (0, 0, 0)), (3, (0, 10, 5)), (3, (19, 29, 24)), (3, (20, 30, 25))\},$$

where the first demand-price pair is due to the initial bid, the second is his or her bid revision in iteration 1, and the remaining records his or her demand at the end of price-increase phases (when he or she is active).

We can write a collection of informative events for bidder i as

$$E_i = \{v_{ij} - p_j \geq v_{il} - p_l, \forall l \in \mathcal{J}, \forall (j, \mathbf{p}) \in C_i\}. \quad (6)$$

5.3. Calculating Privacy Loss

In dynamic auctions such as ours, because of the randomness in the auction path, information revealed could vary from one realization of the auction to another. Thus, the information revealed is a random variable. It is not practical for us to analytically derive the expected amount of information revealed for a given auction mechanism. Instead, we use a numerical simulation approach to approximate bidder information revelation.

By the Bayesian rule, the posterior probability mass function of a bidder's valuation \mathbf{v} is

$$P(\mathbf{v}|E) = \frac{P(E|\mathbf{v})P(\mathbf{v})}{P(E)},$$

where $P(\mathbf{v})$ is given by the prior, and $P(E|\mathbf{v})$ is equal to 1 if \mathbf{v} satisfies all the inequalities in Equation (6) and 0 otherwise. For a set of uniform and independent valuation distributions, the amount of information revealed is simply

$$\begin{aligned} R(\mathbf{V}; E) &= \sum_{\mathbf{v}} [P(\mathbf{v}|E) \log_2 P(\mathbf{v}|E) - P(\mathbf{v}) \log_2 P(\mathbf{v})] \\ &= \sum_{\mathbf{v} \in E} \frac{P(\mathbf{v})}{P(E)} \log_2 \frac{P(\mathbf{v})}{P(E)} - \sum_{\mathbf{v}} P(\mathbf{v}) \log_2 P(\mathbf{v}) \\ &= \log_2 \frac{P(\mathbf{v})}{P(E)} - \log_2 P(\mathbf{v}) \\ &= -\log_2 P(E). \end{aligned}$$

The challenge lies in computing $P(E)$, which amounts to a high-dimension integration problem over a convex feasible region, as defined by constraints in Equation (6). This integration problem is known to be difficult even with numerical integration. We thus resort to a Monte Carlo (MC) simulation approach instead.

The basic idea of the MC simulation method is as follows. We draw N random bidder valuations $\{\mathbf{v}\}$ according to the prior distribution. We test each valuation against the set of constraints in Equation (6).

If all constraints are satisfied, we record 1; otherwise, 0. Suppose that we record N_1 ones; then $P(E)$ can be approximated by N_1/N . Therefore, the amount of information revealed is approximated by

$$R(V; E) \approx \log_2 N - \log_2 N_1. \quad (7)$$

Although Equation (7) provides a straightforward way to simulate the entropy reduction, it would, however, require lots of draws (say, more than 20,000) to obtain a reliable estimate, largely because the MC method is slow in convergence. To accelerate convergence, we adopt the following two strategies.

First, we use a *hybrid approach* that combines numerical simulation and analytical evaluation. We note that whenever there are equal constraints such as $v_{i1} - v_{i2} = 3$, a degree of freedom is lost, and a lot of draws are required to satisfy the equality constraint, which results in slow convergence. To speed things up, we decompose the value vector into free and derived variables. By construction, the derived variables can be mathematically derived using a system of linear equations. The entropy reduction for derived variables can be evaluated analytically without simulation, and the entropy reduction for the free variables will be simulated.

Second, we use a *quasi-MC method* instead of the regular MC method. The former is popular in computational finance for calculating high-dimensional integration (Boyle et al. 1997). The quasi-MC method works the same way as regular MC simulation, but the points of integration are given by a deterministic low-discrepancy sequence rather than the usual pseudorandom sequence. We use the popular Sobol sequence, which is known to achieve a high convergence rate at high dimensions compared with the pseudorandom sequence (Morokoff and Caflisch 1995).

Now we describe an algorithm for calculating the bidder information revelation for ascending clock auctions.

Algorithm 1: A Hybrid Quasi-MC Algorithm for Calculating Information Revelation

1. For the given bidder, given a set of informative events E , extract equality constraints, and use them to decompose valuation variables into a set of free variables F and a set of derived variables D such that the value of any derived variable $v \in D$ can be derived from free variables F via a system of linear equations, let $n' \equiv ||F||$ denote the number of free variables.
2. Randomly draw N size- n' vectors according to the prior distribution.
3. $N_1 \leftarrow 0$.
4. For each set of free variables F , derive the value of derived variables D by solving a system of linear equations. Test whether the free variables satisfy

constraints in E and the lower/upper bounds and whether the derived variables satisfy lower/upper bounds. If all constraints are satisfied, $N_1 \leftarrow N_1 + 1$.

5. Analytically calculate $R(D; E)$ as the maximal information revealed for the derived variables.

6. $\hat{R}(V; E) \leftarrow \log_2(N) - \log_2(N_1) + R(D; E)$.

5.4. Numerical Experiments and Results

In this section, we describe the results of four numerical experiments on the PAC auction. The first three experiments evaluate the privacy-preservation property of the PAC auction against two benchmarks with similar theoretical properties: the DGS auction (DGS for short) and the multi-item VCG mechanism (VCG for short). VCG provides a full revelation benchmark, whereas DGS is a natural benchmark among dynamic auctions for assignment problems. The first two experiments compare PAC against benchmarks in low- and high-competition scenarios. The third experiment examines the role of valuation dispersion—the number of distinct values $\{v_{ij}\}$ can take. The fourth experiment examines the scalability of the PAC auction.

For the DGS auction, we adopt the implementation of Sankaran (1994), which modifies the original DGS to make it more tractable. Following Sankaran (1994), we use the labeling algorithm of Ford and Fulkerson (1962) for finding a set of overdemand items to increase prices. Entropy reduction for DGS auctions can be calculated in a similar way as for PAC auctions, noting that each reported demand set can be seen as a set of bids. The VCG allocation and payments can be solved as integer programming problems. We use the open-source GLPK solver for large-scale mixed-integer programming problems to implement VCG. Because participants of VCG reveal all private information through truthful reporting, we derive the entropy reduction for VCG analytically without simulation.

For the first three benchmarking experiments, we capture entropy reduction along with three other metrics: *number of iterations*, *running time*, and *residual range*. An iteration for DGS, similar to an iteration for PAC, is a single reassignment and price-increase cycle. The running time refers to the actual time used for solving the assignment problem. It does not include the time used for computing the entropy reduction, which is done after the auction/mechanism.

Residual range is a simplified measure of information revelation based on the intuition that information revelation can reduce the range of possible valuations. From the discussion in Section 5.2, the information revelation introduces new inequality constraints in the form of Equation (6). Though we cannot derive a precise range of item valuations

v_{ij} , we can, however, calculate the range of $v_{ij} - v_{ik}$. Specifically, denoting $[a_{ijk}, b_{ijk}]$ and $[a'_{ijk}, b'_{ijk}]$ as the prior and posterior ranges of $v_{ij} - v_{ik}$, respectively, we define the *residual range* for bidder i , RR_i , as

$$RR_i = \frac{\sum_{j,k \in \phi} (b'_{ijk} - a'_{ijk})}{\sum_{j,k \in \phi} (b_{ijk} - a_{ijk})}, \text{ where } \phi = \{j, k | j, k \in \mathcal{J}, j < k\}. \quad (8)$$

Intuitively, RR_i is the sum of posterior ranges divided by the sum of prior ranges for all unique $\{j, k\}$ pairs. Because the ranges do not increase, the residual range is always between 0 and 1. The higher the residual range, the less information gets revealed by the auction. For example, suppose that valuations can take integer values from 0 to 15, and the prior and posterior ranges of $v_{11} - v_{12}$ are $[-15, 15]$ and $[0, 10]$, respectively. The residual range for this pair of valuations is $11/31$.

In all four experiments, we assume that bidder values are independently and uniformly distributed. We set the value of outside options and auctioneer reservation values to 0 for simplicity. Unless otherwise noted, we draw integer bidder values from a uniform distribution between 0 and 15 (see Experiment 3 for alternative cases). We use 5,000 draws for all quasi-MC simulations because our experimentation shows that this is a reasonable number for achieving convergence. All experiments are carried out using R on a desktop computer with 16 GB of random-access memory and an Intel Core i3-4130 central processing unit.

5.4.1. Experiment 1: Benchmarking Tests with Low Competition. In the first experiment, we benchmark PAC against DGS and VCG in a low-competition environment, where we let the number of items be the number of bidders minus 1. We systematically vary the total number of items from 2 to 20 in increments

of 2. For each problem size, we simulate 50 scenarios, each corresponding to a set of bidder valuations. We run three mechanisms on each of the scenarios and report the average outcome metrics (i.e., entropy reduction, residual range, number of iterations, and running time) across 50 scenarios in Table 3.

As we can see from Table 3, our PAC auction consistently outperforms DGS on privacy preservation: it results in 6.2%–18% less entropy reduction than DGS, with its advantage more significant at larger problem sizes. This increasing benefit of PAC makes intuitive sense: with increasing numbers of bidders and items, it is increasingly unnecessary for bidders to reveal their full demand sets to find an optimal allocation. The residual range for PAC is consistently higher than those for DGS, suggesting that PAC is better at preserving bidder privacy. Both PAC and DGS have a significant privacy-preservation advantage over VCG, confirming the informal argument in the auction literature that dynamic auctions are better for protecting bidder privacy (Ausubel 2006, Ausubel and Milgrom 2002, de Vries et al. 2007, Lucking-Reiley 2000). For example, PAC results in 46%–86% less entropy reduction than VCG.

Our PAC auction uses a comparable number of iterations as DGS in many cases, but it seems that PAC uses fewer iterations for larger problems (which we confirm in later experiments). The running times of PAC are 1.7–2.3 times that of DGS. Unsurprisingly, both dynamic auctions are significantly slower than VCG.

5.4.2. Experiment 2: Benchmarking Tests with High Competition. The second experiment is similar to the first, except that we now benchmark PAC against DGS and VCG in a high-competition environment, where we keep the number of items approximately at one-half the number of bidders. As shown in Table 4, the number of iterations is noticeably higher for both PAC and DGS, which makes sense because of

Table 3. Experiment 1: Benchmarking Tests with Low Competition

n	m	Entropy reduction				Residual range			Iterations			Running time (sec)			
		PAC	DGS	VCG	t	PAC	DGS	t	PAC	DGS	t	PAC	DGS	VCG	t
3	2	13.1	13.9	24	-3.6***	0.43	0.38	7.7***	3.0	3.0	NA	0.07	0.04	0.006	3.8***
5	4	32.2	36.2	80	-7.8***	0.60	0.58	7.3***	5.2	5.3	-1.5	0.19	0.11	0.004	11.4***
7	6	53.9	60.0	168	-7.2***	0.69	0.66	9.9***	6.5	6.5	-0.5	0.39	0.19	0.008	10.8***
9	8	76.5	85.4	288	-10.3***	0.74	0.71	9.6***	7.8	7.9	-0.7	0.64	0.31	0.011	13.1***
11	10	100.6	114.0	440	-11.9***	0.79	0.76	12.1***	8.3	8.2	0.3	0.95	0.44	0.012	13.7***
13	12	130.2	148.4	624	-13.6***	0.82	0.79	14.1***	9.2	8.9	1.7	1.30	0.58	0.019	15.2***
15	14	153.4	180.1	840	-15.6***	0.84	0.81	16.4***	9.2	9.1	0.5	1.64	0.73	0.028	20.9***
17	16	182.0	209.3	1,088	-12.9***	0.85	0.83	18.3***	9.5	9.3	0.7	2.08	0.92	0.035	22.4***
19	18	203.6	238.9	1,368	-17.4***	0.86	0.84	18.7***	9.3	9.3	0.1	2.50	1.10	0.050	21.5***
21	20	240.6	293.4	1,680	-23.5***	0.88	0.85	21.7***	9.6	10.2	-2.8**	3.47	1.52	0.069	18.8***

Notes. Paired t -tests comparing PAC and DGS auctions.
 ** $p < 0.01$; *** $p < 0.001$.

Table 4. Experiment 2: Benchmarking Tests with High Competition

n	m	Entropy reduction				Residual range			Iterations			Running time (sec)			
		PAC	DGS	VCG	t	PAC	DGS	T	PAC	DGS	t	PAC	DGS	VCG	t
3	2	13.3	14.2	24	-3.9***	0.43	0.41	5.3***	3.0	3.0	NA	0.07	0.04	0.007	3.2**
5	3	31.6	34.2	60	-4.9***	0.48	0.45	4.9***	5.7	5.9	-1.0	0.17	0.12	0.005	8.2***
7	4	54.4	60.9	112	-6.9***	0.56	0.53	4.3***	8.3	8.3	0.1	0.36	0.22	0.006	12.0***
9	5	77.1	86.3	180	-6.7***	0.63	0.59	5.6***	10.1	10.4	-0.7	0.59	0.35	0.009	11.6***
11	6	102.4	116.2	264	-7.1***	0.67	0.64	6.8***	12.9	12.4	1.1	0.89	0.50	0.014	13.8***
13	7	124.9	149.6	364	-10.3***	0.71	0.66	9.5***	13.2	14.5	-2.1*	1.18	0.69	0.014	10.3***
15	8	149.2	178.0	480	-11.8***	0.74	0.70	12.0***	14.4	16.3	-3.8***	1.63	0.94	0.022	10.7***
17	9	176.4	210.1	612	-10.2***	0.77	0.72	10.6***	15.3	17.0	-2.8**	1.91	1.13	0.023	11.5***
19	10	202.4	240.1	760	-13.8***	0.79	0.74	13.6***	15.3	17.0	-3.0**	2.40	1.36	0.031	14.6***
21	11	230.5	276.8	924	-13.1***	0.80	0.76	16.3***	16.6	18.4	-2.9**	3.06	1.65	0.042	13.7***

Notes. Paired *t*-tests comparing PAC and DGS auctions.

p* < 0.05; *p* < 0.01; ****p* < 0.001.

intensified competition. Still, PAC consistently outperforms DGS in privacy preservation, with 6.3%–16.7% less entropy reduction and a bigger advantage at larger problem sizes. The privacy-preservation advantage of dynamic auctions (PAC and DGS) is less than the first experiment—PAC results in 44%–75% less entropy reduction than DGS, which makes intuitive sense because bidders are forced to reveal more information with intensified competition. It becomes more apparent that PAC uses fewer iterations than DGS for larger problems. The running time for PAC is 1.6–1.8 times that of DGS. Again, in this scenario, the residual range of PAC is also consistently higher than DGS, confirming the former's privacy-preservation advantage.

5.4.3. Experiment 3: Benchmarking Tests with Different Valuation Dispersion. In the third experiment (Table 5), we examine the role of valuation dispersion. We fix the problem size to five items and nine bidders but vary the support of the value distribution, with each step doubling the support of the previous step, thus increasing the value dispersion. As the support increases, both dynamic auctions use more iterations and a longer running time. Again, PAC consistently

outperforms DGS in privacy preservation, resulting in 2.9%–17.5% less entropy reduction. The privacy-preservation advantage of PAC is more prominent when the support is smaller (valuation dispersion is low). This is because the privacy-preservation advantage of our PAC auction is more pronounced when the demand set is large, which is more likely when the valuations for different items are less dispersed (i.e., the support is smaller). Interestingly, as the support increases, the running time of PAC becomes closer to that of DGS, from 2.5–1.4 times. This is consistent with our intuition that there are fewer cases of large demand sets as the support increases, reducing the difference between partial and full reporting. The finding using residual range is similar, with the difference more significant when the support is smaller.

5.4.4. Experiment 4: Scalability Test. Our fourth experiment examines how our PAC auction scales as the problem size increases and benchmarks PAC against DGS and VCG for large-scale problems in terms of iterations and running times. In this experiment, we study a series of nine problem sizes, from 2×1 (two bidders and one item) to 512×256 , with each step

Table 5. Experiment 3: Benchmarking Tests with Different Support

Support	Entropy reduction				Residual range			Iterations			Running time (sec)			
	PAC	DGS	VCG	t	PAC	DGS	t	PAC	DGS	t	PAC	DGS	VCG	t
0, 3	49.0	59.4	90.0	-12.1***	0.66	0.54	16.0***	4.4	4.9	-4.1***	0.58	0.23	0.011	17.6***
0, 7	66.2	76.9	135.0	-9.0***	0.64	0.57	11.2***	8.1	7.9	0.6	0.64	0.32	0.008	14.0***
0, 15	78.3	89.0	180.0	-8.5***	0.63	0.58	8.3***	10.9	11.2	-0.5	0.77	0.46	0.012	6.1***
0, 31	90.9	98.7	225.0	-4.3***	0.62	0.59	4.2***	13.2	13.2	0.1	0.84	0.53	0.009	7.6***
0, 63	101.8	106.3	270.0	-2.4*	0.62	0.59	3.2**	13.5	13.7	-0.3	0.85	0.56	0.011	6.2***
0, 127	115.3	124.4	315.0	-4.2***	0.61	0.60	1.8	14.5	15.0	-1.1	0.87	0.58	0.011	5.8***
0, 255	126.9	130.6	360.0	-1.5	0.61	0.60	2.0*	14.5	14.6	-0.4	0.90	0.62	0.011	5.7***

Notes. Paired *t*-tests comparing PAC and DGS auctions.

p* < 0.05; *p* < 0.01; ****p* < 0.001.

doubling the numbers of bidders and items in the previous step. We consider the largest problem, which has 132,072 (= 512 × 256) decision variables, a stress test of the PAC auction’s scalability. For each problem size, we generate 10 random scenarios, conduct a PAC auction for each scenario, and report the average outcome metrics in Table 6.¹⁶ We also run DGS and VCG on the same scenarios and record their running times and number of iterations (when applicable). Besides the usual metrics of iterations and running times, we also track the *total bid revisions* and the *total inquiries* for PAC, as described in the PAC auction example. These help us understand the amount of activity by the bidders and the auctioneer.

We visualize the PAC results in Figure 2. To accommodate a large range of problem sizes, we use logarithmic scales for both axes. In such a log–log plot, a linear line represents a power-law relationship, with $\log(y) = \log(a) + b \log(x)$ translating to $y = ax^b$.

As we can see from Figure 2 and Table 6, as the number of variables ($m \times n$) increases, the number of iterations first increases and then decreases. In fact, for the largest problem, the average number of iterations is only 2.1. This is because when the numbers of bidders and items are both high, the chances of two bidders “clashing” over the same item is low because each bidder has many choices to choose from. As a result, the PAC algorithm spends most of the time rearranging bidders rather than increasing prices.

The number of total bid revisions, the number of total inquiries, and running time all have a power-law relationship with the number of variables. The number of total bid revisions ($b = 0.8$) shows a decelerating trend, whereas the running time ($b = 1.1$) shows a slightly accelerating trend. The number of total inquiries ($b = 0.96$) is approximately linear in the number of variables. We note that the number of bid revisions is far less than the theoretical limit. For the three largest problems, the number of *total bid revisions* is 6.7%–10.5% of the number of variables—the

theoretical upper limit for *each iteration*; the number of *total inquiries* is 34%–57% of the number of variables. The total running time for the largest problem with 131,072 variables was about 4 hours. In contrast, DGS was unable to solve problems of 256×128 or larger under 10 hours—we had to terminate DGS prematurely. This is likely because the PAC auction is more efficient in finding an overdemand set by maintaining a provisional assignment. In terms of running times, the gap between PAC and VCG becomes narrower for larger problems. For instance, for the 512×256 problems, VCG used 6,172 seconds on average, whereas our PAC auction used 14,580 seconds, merely 2.2 times the VCG running time. These results suggest that our PAC auction is scalable to a problem size of over 100,000 variables.

6. Conclusion

Motivated by bidders’ privacy concerns, we have proposed (1) an ascending clock auction for unit-demand assignment problems that economizes on bidder information revelation, (2) a new, general-purpose measure of bidder information revelation for auctions and other mechanisms based on Shannon’s entropy, and (3) a simulation-based numerical procedure for computing this measure for ascending clock auctions.

The key departure of our auction design is that it leverages partial reporting without sacrificing important economic properties such as sincere bidding. This partial reporting design leads to less information revelation. Our numerical simulations show that our auction consistently outperforms a full-reporting benchmark by up to 18% savings in the amount of information revealed. Our results show that the informational savings of PAC auctions is more significant for larger problems. This is because in larger problems, bidders only need to reveal a small proportion of their valuations to achieve a successful allocation.

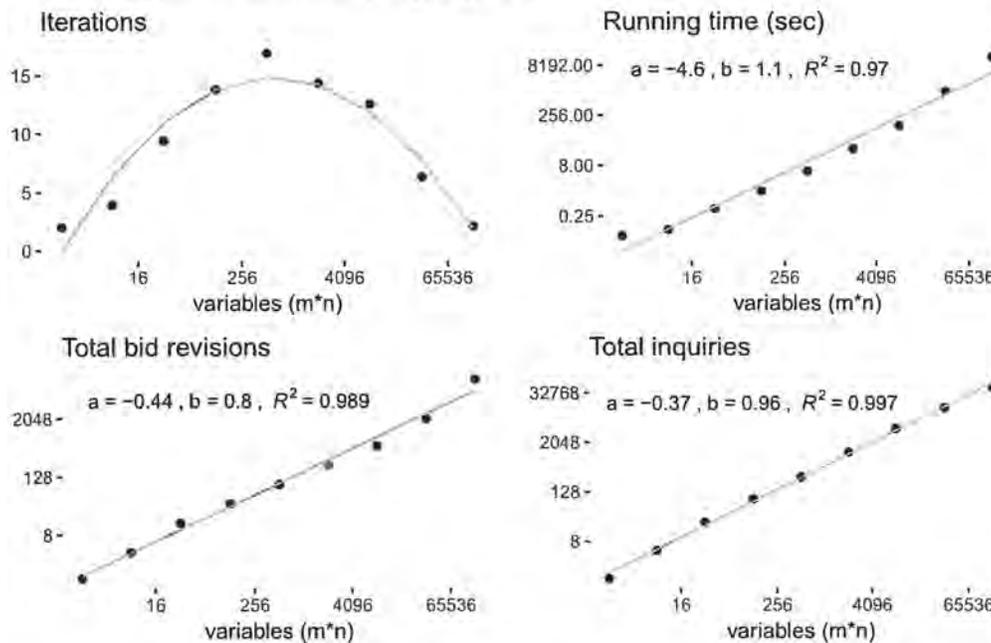
Table 6. Experiment 4: Scalability Test

<i>m</i>	<i>n</i>	<i>m</i> × <i>n</i>	PAC		Iterations			Running time (sec)			
			Total bid revisions	Total inquiries	PAC	DGS	<i>T</i>	PAC	DGS	VCG	<i>t</i>
2	1	2	1.0	1.0	2.0	2.0	NA	0.06	0.04	0.01	1.3
4	2	8	3.5	4.8	3.9	4.4	−1.9	0.10	0.11	0.01	−1.3
8	4	32	14.4	23.5	9.4	9.1	0.4	0.41	0.35	0.02	2.4
16	8	128	36.3	88.6	13.8	16.6	−2.1	1.39	1.26	0.04	1.2
32	16	512	92.2	305.7	16.9	20.3	−2.4*	5.58	3.59	0.25	5.4***
64	32	2,048	233.3	1,243.7	14.3	26.0	−6.5***	25.29	16.51	1.08	8.2***
128	64	8,192	581.2	4,694.6	12.5	26.7	−7.7***	124.90	229.21	27.18	−1.8
256	128	32,768	2,203.4	14,771.9	6.3	NA	NA	1,371.00	NA	343.50	NA
512	256	131,072	13,780.2	44,950.1	2.1	NA	NA	14,580.00	NA	6,712.00	NA

Notes. Paired *t*-tests comparing PAC and DGS auctions.

p* < 0.05; **p* < 0.001.

Figure 2. (Color online) Experiment 4: Scalability Log-Log Plots



Our examples suggest that information savings occur because only one of the tying marginal bidders is required to submit a new bid and because a bidder is only required to reveal one of several preferred alternatives. Information savings may also come from different price paths followed by PAC auctions, though further investigation is required to assess the contribution of each source.

We contribute to the dynamic auction literature by demonstrating the possibility of economizing on bidder information revelation while maintaining desirable economic properties. In our auction, sincere bidding is an ex post Nash equilibrium, and ending prices depend only on starting prices. Efficient allocation is a special case of our auction when starting prices are set to be the reservation values of the auctioneer. Our auction is sincere even when it starts with inefficient prices. This is important because sellers often pursue revenue maximization at the cost of efficiency.

Our entropy-based measure of information revelation and the associated computational framework lay the ground work for more research on privacy preservation in mechanism design. The measure compares the amount of information in the prior and posterior of a bidder's valuations. It is general enough to be used in different mechanisms—including auction and nonauction mechanisms and one-shot and dynamic mechanisms—so that we can compare their ability to preserve participant privacy. There have been many informal remarks about privacy-preservation properties of different mechanisms (e.g., second- versus

first-price auctions, uniform- versus discriminative-price multiunit auctions). We hope this measurement paves the way for formally and systematically studying this important dimension of mechanism design.

The PAC auction has a number of properties that make it attractive for practical use. First, bidding is more natural in our auction than in full-reporting designs. In our auction, bidders, when asked, only need to decide whether to stay or leave. When they decide to leave, they only need to name on a single alternative as their new bid. This type of simple bids is intuitive and easy to formulate. Second, the PAC auction maintains a provisional assignment for bidders and updates it when there are excessive demands. This design is more transparent for bidders than DGS and other primal-dual designs that may leave some bidders unassigned during the auction. Third, though the PAC uses more queries, the amount of work for each individual bidder is quite manageable, because not all bidders are asked in each query, and when they are, their decisions are simple. For example, in our simulation for the 128 (items) \times 64 (bidders) problems, on average each bidder needs to submit nine bids during the entire auction.

Finally, the PAC auction proves to be quite scalable. Our simulations suggest that it can handle assignment problems with over 100,000 decision variables. Our initial evidence suggests that it outperforms the DGS auction for large problems in terms of running time. When the numbers of bidders and items are large, it may be more optimal to delegate bidding activities to an autonomous software agent that acts

on behalf of the bidder. The privacy-preservation benefit of our auction still applies in such “proxy” auctions since the software agent would also reveal less information about the bidder.

There are many directions for extending the current work. First, it would be useful to conduct further analysis on the complexity and scalability of the PAC auction and pin down the sources of its informational advantages, given our promising initial results. Second, future research could optimize the heuristics for choosing auction paths, which could lead to further improvements on bidder privacy and speed. Third, we have provided a proof-of-concept procedure for calculating the entropy-based measure of information revelation. There is still room for optimizing this procedure and adapting it for more complex use cases. Fourth, we believe that much can be gained by comparing other auction formats (e.g., first- versus second-price auctions) using our proposed information-revelation measure. Fifth, we have so far limited ourselves to the amount of information revealed. Our reporting design undoubtedly also impacts the amount of communication and the computational complexity of the winner determination problem, which are good candidates for a next step. Finally, it will be interesting to see how our dynamic auction design could be extended to more complex assignment problems, such as multiunit and combinatorial assignments, and cases where bidder preferences may shift at different prices. We conjecture that with the increasing complexity of the demand space, *as-needed* partial reporting has even higher value.

Appendix. Proofs

A.1. Proof of Lemma 1

We first argue that when all bidders are sincere, then the auction must stop in finite iterations. Suppose that it does not. Because at any t at least one price should increase, as $t \rightarrow \infty$, there must exist one item j whose price is high enough such that no sincere bidder would bid on this item. But the price of an unassigned item must equal its starting price (Observation 3), a contradiction. Now we consider that all but one bidder bid sincerely. Again, for a large enough t , at least one item has a price high enough and it exceeds all bidders’ valuations. Because the item must be assigned and any sincere bidder must have left the item already, the only remaining bidder on this item must be the insincere one. But with just one bidder on this item, the price of this item will never rise again. So the auction of the rest of the items, in which only the sincere bidders participate, must end in finite iterations. \square

A.2. Proof of Lemma 2

We discuss two cases: (a) 1 is assigned to the same item, say j , in both outcomes, and (b) 1 is not. In case (a), we let $\mathbf{I} = 1$. Because 1 strictly prefers (x', p') to (x, p) , we clearly have $p'_j < p_j$. We now consider case (b). Suppose that 1 is assigned to j_1 in x and j_2 in x' ($j_1 \neq j_2$). Because bidder 1 strictly prefers

(x', p') to (x, p) and j_1 is a sincere bid at p , we have $v_{1j_1} - p_{1j_1} < v_{1j_2} - p'_{j_2}$ and $v_{1j_2} - p_{j_2} \leq v_{1j_1} - p_{1j_1}$, which implies that

$$p'_{j_2} < p_{j_2}. \quad (\text{A.1})$$

By Observation 3, j_2 must be assigned in (x, p) because $p_{j_2} > p'_{j_2} \geq p_{j_2}^0$. Let C be the winner of item j_2 in (x, p) . Because the price of j_2 is lower in (x', p') , C must not drop out and be a winner of some item j_3 in (x', p') . The fact that C bids sincerely at p and p' (C is not 1) implies that $v_{Cj_2} - p_{j_2} \geq v_{Cj_3} - p_{j_3}$ and $v_{Cj_3} - p'_{j_3} \geq v_{Cj_2} - p'_{j_2}$. So

$$p'_{j_2} - p'_{j_3} \geq v_{Cj_2} - v_{Cj_3} \geq p_{j_2} - p_{j_3} \Rightarrow p'_{j_3} - p'_{j_2} \leq p_{j_3} - p_{j_2}.$$

Combining this with Equation (A.1), we have

$$p'_{j_3} < p_{j_3}. \quad (\text{A.2})$$

Similarly, we can find another bidder who is the winner of j_3 in (x, p) and a winner of another item at (x', p') . Because we have a finite number of items, we must end up with a bidder who is a winner of item j in (x, p) —thus, we have a permutation. By construction, every item in this permutation has a lower final price in (x', p') than in (x, p) . \square

A.3. Proof of Lemma 3

We prove Lemma 3 by contradiction. Suppose that the two outcomes result from auctions A and B , respectively, and bidder 1 strictly prefers (x', p') to (x, p) . By Lemma 2, there must be a subset of bidders \mathbf{I} such that $1 \in \mathbf{I}$, $x'(\mathbf{I})$ is a permutation of $\mathbf{I}(\mathbf{I})$, and $p'_j < p_j, \forall j \in x(\mathbf{I})$.

Case 1. Suppose that \mathbf{I} includes all the assigned bidders in x . Consider the last reassignment involving items $x(\mathbf{I})$ in auction A . There are only two possible subcases:

(a) A bidder i chooses the outside option 0 over item $j \in x(\mathbf{I})$ at price p_j and all other items at their respective prices. Because bidder i bids sincerely, i is indifferent between j and 0 at this price, which means i strictly prefers j to 0 when j 's price is $p'_j < p_j$. Since i is not a winner of any item in $x'(\mathbf{I})$ in auction B and he prefers j to 0 under p' , he must be a winner of another item, say $k \notin x'(\mathbf{I})$, and strictly prefer k to 0 under p' . This contradicts the fact that i chooses 0 over k in auction A when $p_k = p_k^0 \leq p'_k$.

(b) The last reassignment involves a bidder i moving into an unassigned item $j \in x(\mathbf{I})$. By Observation 3, item j must still have a price of $p_j = p_j^0$, contradicting $p_j > p'_j \geq p_j^0$.

Case 2. Suppose that \mathbf{I} does not include all the assigned bidders in (x, p) . Consider the last reassignment involving items $x(\mathbf{I})$ in auction A . By (b), the last reassignment cannot be some bidder moving into an unassigned item in $x(\mathbf{I})$. The only other possibility is a bidder leaving $x(\mathbf{I})$. Suppose that bidder i is the last bidder leaving $x(\mathbf{I})$. Suppose further that he or she leaves item $j_1 \in x(\mathbf{I})$ at time t of the auction when j 's price is p_{j_1} . Because $p'_{j_1} < p_{j_1}$, i must be a winner of some item under p' , say item j_2 . By sincere bidding of i (recall that $1 \in x(\mathbf{I})$ and $i \notin x(\mathbf{I})$) at two auctions, we have $v_{ij_2} - p'_{j_2} \leq v_{ij_1} - p'_{j_1}$ and $v_{ij_1} - p_{j_1} \geq v_{ij_2} - p'_{j_2}$, respectively. Because $p'_{j_1} < p_{j_1}$ and $p'_{j_2} \leq p_{j_2}$ (no price drop), we can infer

$$p'_{j_2} < p_{j_2}. \quad (\text{A.3})$$

From this point on we can repeat the argument of Lemma 2 and obtain a new set of bidders $I_1 = I \cup \Delta$, $i \in \Delta$, such that $x'(I_1)$ is a permutation of $x(I_1)$ and $p'_j < p_j, \forall j \in x(I_1)$. By induction and the fact that there are a limited number of bidders, we must end with case 1—a contradiction. \square

A.4. Proof of Proposition 1

Immediate from Lemma 3. \square

A.5. Proof of Proposition 2

We reformulate the assignment problem by replacing auctioneer 0 with m dummy agents $\{n+1, \dots, n+m\}$ and the outside option 0 with n dummy items $\{m+1, \dots, m+n\}$. With the reformulated assignment problem, any feasible assignment can be represented by a permutation of bidders (including dummies).

Consider a sincere auction outcome $(x, p) \in \Omega$. We can easily verify that when the starting prices are set to the auctioneer's reservation valuations, the condition

$$x_i \in D_i(p), i \in \{1, 2, \dots, m, m+1, m+2, \dots, m+n\} \quad (\text{A.4})$$

holds for both real and dummy bidders. In other words, every bidder, including dummy ones, weakly prefers his or her assignment.

We now claim that any cyclic permutation of x weakly decreases efficiency. We suppose that i_1, i_2, \dots, i_l are originally assigned to items j_1, j_2, \dots, j_l , but a cyclic permutation reassigns them to $j_2, j_3, \dots, j_l, j_1$. By condition (A.4), we know that

$$\begin{aligned} v_{i_1 j_1} - p_{j_1} &\geq v_{i_1 j_2} - p_{j_2}, \\ v_{i_2 j_2} - p_{j_2} &\geq v_{i_2 j_3} - p_{j_3}, \\ &\dots, \\ v_{i_l j_l} - p_{j_l} &\geq v_{i_l j_1} - p_{j_1}. \end{aligned}$$

Adding two sides of inequations, we have

$$\sum_{k=1..l} v_{i_k j_k} \geq \sum_{k=1..l-1} v_{i_k j_{k+1}} + v_{i_l j_1}.$$

The left (right)-hand side is the efficiency of the original (permuted) assignment, implying that the cyclic permutation weakly decreases efficiency. Because any permutation can be decomposed into several disjoint cyclic permutations, we conclude that any feasible assignment other than x weakly decreases efficiency. \square

A.6. Proof of Proposition 3

Consider an auction outcome (x, p) . Without loss of generality, we assume that bidder 1 is assigned to $j_1 \in \mathcal{F}$ under x . We consider an alternative economy where bidder 1 is excluded from participation. The VCG payment for bidder 1 can thus be calculated as the difference in social welfare of all other players in the original and alternative economies. We compare auction payments and VCG payments for two cases: (a) $p_{j_1} > p_{j_1}^0$ and (b) $p_{j_1} = p_{j_1}^0$.

Case 1. We construct a new, efficient assignment \tilde{x} in the alternative economy and use it to calculate the VCG payment for bidder 1. There must be a last bidder who leaves j_1 at price p_1 . Suppose that this bidder is i_2 and assigned to $j_2 \in \mathcal{F}$ under

x . If $p_{j_2} > p_{j_2}^0$, we continue to search for the last bidder who leaves j_2 , and so on. Eventually we can find a series of bidders $\{i_2, i_3, \dots, i_k\}$ assigned to $\{j_2, j_3, \dots, j_k\}$ such that i_l is the last bidder to leave j_{l-1} , for $l = 2 \dots k$, and the price of item j_l is strictly above its starting price except $p_{j_k} = p_{j_k}^0$. We now construct a new assignment \tilde{x} under which every bidder in the series $\{i_2, i_3, \dots, i_k\}$ takes the place of the preceding bidder (e.g., bidder i_2 is assigned to item j_1), i_1 is unassigned, and all other bidders keep their assignments. For bidder i_2 , we know by construction that (t is the moment of the last bidder departure at j_1)

$$v_{i_2 j_1} - p_{j_1} \geq v_{i_2 j_2} - p_{j_2}^t \geq v_{i_2 j_2} - p_{j_2}, \quad (\text{A.5})$$

where the first inequality is by sincere bidding and the second inequality is because of nondecreasing prices. Hence, given p , i_2 would prefer his or her assignment j_1 . The same argument can be made for other bidders in the series $\{i_2, i_3, \dots, i_k\}$. Overall, we conclude that $\tilde{x}_i \in D_i(p)$ holds for all bidders, and by Proposition 2, \tilde{x} is efficient in the alternative economy.

By sincere bidding in the original economy, we also have $v_{i_2 j_1} - p_{j_1} \leq v_{i_2 j_2} - p_{j_2}$. Combining this with (A.5), we have

$$v_{i_2 j_1} - p_{j_1} = v_{i_2 j_2} - p_{j_2}. \quad (\text{A.6})$$

Equation (A.6) is also true for other bidders in the series. Summing up all equations and rearranging terms, we have

$$(v_{i_2 j_1} + v_{i_3 j_2} + \dots + v_{i_k j_{k-1}} + p_{j_k}) - (v_{i_2 j_2} + v_{i_3 j_3} + \dots + v_{i_k j_k}) = p_{j_1}.$$

Note that $p_{j_k} = p_{j_k}^0 = v_{0 j_k}$, so the first parentheses represent the social welfare of bidders $\{i_2, i_3, \dots, i_k\}$ and the auctioneer in the alternative economy, and the second parentheses, the original economy. So the left-hand side is exactly the VCG payment of bidder 1 for assignment x , suggesting that the auction payment p_{j_1} for bidder 1 coincides with his VCG payment for the same assignment. \square

A.7. Proof of Lemma 4

Because $(x, p) \in \Omega$, by Lemma 3, bidder i weakly prefers (x, p) to any outcome in Ω_{-i} . Because $\Omega \subseteq \Omega_{-i}$, the bidder must also weakly prefer (x, p) to any other $(x', p') \in \Omega$. To see that p is indeed the smallest price vector in Ω , we discuss two cases. (a) If bidder i is assigned to the same item in x and x' , $p_{x_i} \leq p'_{x_i}$ holds trivially. Otherwise, say, (b) i is assigned to item 1 under x but 2 under x' . We have

$$\begin{aligned} v_{i,2} - p'_2 &\geq v_{i,1} - p'_1 \\ v_{i,1} - p_1 &\geq v_{i,2} - p'_2, \end{aligned}$$

where the first inequality is because item 2 must be weakly preferred by i under (x', p') and the second inequality is because i weakly prefers (x, p) to (x', p') . These two inequalities imply that $p_1 \leq p'_1$. Hence, any assigned item must have the lowest price in Ω . If the item is unassigned, its price is the starting price, which is the lowest possible price in Ω . Overall, we conclude that p is the lowest price in Ω . \square

A.8. Proof of Proposition 4

The proof of path independence follows naturally from Lemma 4, which shows that all auction paths must end with

the same smallest price. To see that identical items have the same ending price, we suppose two items 1 and 2 have different final prices. Without loss of generality, we assume that $p_1 > p_2$. Because their starting prices are the same, 1 must be assigned to a bidder, say, bidder 1. By sincerity, we have $v_{11} - p_1 \geq v_{12} - p_2$. But because $v_{11} = v_{12}$ (because they are identical), we have $p_1 \leq p_2$, a contradiction. \square

Endnotes

¹ The reputation of the auctioneer in the market may engender some level of trust. However, reputation as a tool for addressing privacy concerns of bidders has its own limitations—that is, it may give well-established auctioneers a great advantage over newcomers, which deters entry and negatively impacts the efficiency of the market.

² Several practical auction designs for more complex problems, such as simultaneous multiround auctions and combinatorial clock auctions, do not require full reporting (see chapters 1, 5, 7, 8, and 15 in the *Handbook of Spectrum Auction Design*; Bichler and Goeree 2017). These auctions use activity rules to mitigate insincere bidding. More specifically, the activity rules counter—but may fail to completely overcome—the incentive for a strategic wait-and-see behavior by the bidders. These auction designs are not directly comparable to ours because they are designed for more general demand structures and have quite different designs. Some practical dynamic auction designs for unit-demand assignment problems do not require full reporting, but they do not have the same economic properties as ours (see our literature review).

³ de Vries et al. (2007) briefly discuss how some ascending auctions, such as the auction in (Ausubel and Milgrom 2002), can be interpreted as a result of replacing the primal-dual algorithm—when solving the efficient allocation problem—with a subgradient algorithm. The subgradient algorithm does not require full reporting. However, as noted by de Vries et al. (2007), it needs not converge in finite iterations, and the prices in such auctions can actually decrease.

⁴ A *one-way function* is a function whose value is “easy” to compute for every input but is “hard” to invert, where easy and hard are in the sense of the computational complexity theory. The existence of such functions is an open conjecture.

⁵ If i' is also singularly assigned, find a replacement bidder for i' , and so on, until we have a chain of replacement that ends with a non-singular bidder. We can always find a replacement bidder i' whose revealed demand includes item 1. This is because, by our way of constructing the active set (see step 2e for details), the only way a singular item 1 can enter the active set is through active set expansion, where a marginal bidder j at item 2 is indifferent between item 2 and item 1, and item 1 is added to the active set as a result. The chain of replacement always ends in finite steps because there must be at least one active bidder who is not singular.

⁶ Our results still hold as long as an item with more than one bidder is chosen as an initial active set.

⁷ Again, if a chain of replacement is required, expand the active set to include all items on the chain.

⁸ The auction prices can increase discretely or continuously since our auction design works for both discrete and continuous values.

⁹ We note that the outside option cannot enter the active set via expansion. The argument is as follows. We suppose that it has not entered the active set so far. The only way for it to enter the active set is via active set expansion, which would require that the current marginal bidder has previously been assigned to the outside option and left it—a contradiction.

¹⁰ The DGS auction closely resembles the Hungarian method (Demange et al. 1986). While the Hungarian method finds augmenting paths by computing full demand sets from known valuations, DGS does so by asking bidders to report them.

¹¹ Our proof strategy contrasts with those used in the literature. The literature typically assumes the auctioneer has 0 reservation values and the auction starts with a 0 price vector, then shows that the auction terminates with a price vector that is minimal within a lattice of Walrasian equilibria (Demange et al. 1986, Mishra and Talman 2010, Andersson et al. 2013, Ausubel 2006, de Vries et al. 2007). Showing efficiency and “strategy proofness” (a concept related to sincerity) follow almost immediately. Our auction allows for non-trivial reservation values and arbitrary starting prices, and therefore needs not be efficient. Thus, we have followed a different approach that first establishes sincerity without requiring efficiency then uses sincerity to study auction’s terminal state.

¹² When starting prices are equal to the seller’s reservation values, our definition of Ω reduces to the definition of competitive prices in DGS (1986). When starting prices and reservation values are 0, the elements of Ω are simply the Walrasian equilibria as defined in Gul and Stacchetti (2000).

¹³ We note that in an assignment with more general preferences (i.e., complementary demands), a Walrasian equilibrium may not exist, and even if it exists, the minimum Walrasian price need not coincide with the VCG payments (Gul and Stacchetti 1999, 2000).

¹⁴ For an excellent reference on entropy as well as for an intuitive explanation of why it is an appropriate tool for quantifying information, see chapter 2 of Cover and Thomas 1991.

¹⁵ We are measuring the information revealed by an event *ex post*. This can be viewed as a special case of the extensive literature on measuring, *ex ante*, the expected information revealed by a random variable Y . In this case, the information Y reveals (*ex ante*) about X , denoted by $R(X; Y)$, is the average information revealed by each possible outcome of Y . In other words, it is the sum of $Pr(Y = y)R(X, y)$ over all possible outcomes of Y . The expression $R(X, Y)$ is often referred to in the literature as the mutual information between X and Y .

¹⁶ Because the goal here is scalability, and it is time-consuming to compute the entropy reduction for extremely large problems, we do not compute entropy reduction in this experiment. Also, we keep the number of scenarios to 10 to limit the total running time for large problems.

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