Rare Disasters and Risk Sharing with Heterogeneous Beliefs

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Abstract

Although the threat of rare economic disasters can have large effect on asset prices, difficulty in inference regarding both their likelihood and severity provides the potential for disagreements among investors. Such disagreements lead investors to insure each other against the types of disasters each one fears the most. Due to the highly nonlinear relationship between consumption losses in a disaster and the risk premium, a small amount of risk sharing can significantly attenuate the effect that disaster risk has on the equity premium. We characterize the sensitivity of risk premium to wealth distribution analytically. Our model shows that time variation in the wealth distribution and the amount of disagreement across agents can both lead to significant variation in disaster risk premium. It also highlights the conditions under which disaster risk premium will be large, namely when disagreement across agents is small or when the wealth distribution is highly concentrated in agents fearful of disasters. Finally, the model predicts an inverse U-shaped relationship between the equity premium and the size of the disaster insurance market.

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1 Introduction

How likely is it that a severe economic disaster will occur in the next 100 years? With a relatively short sample of historical data, it is difficult to accurately estimate the likelihood of disasters or the size of their impact. For example, one cannot reject a constant disaster intensity of 3% at the 5% significance level even after observing 100 years without a disaster. This suggests that there is likely to be significant heterogeneity in the beliefs of market participants about disasters. In this paper, we show that such disagreements can generate strong risk sharing motives among investors and significantly affect asset prices.

We study an exchange economy with two types of agents. Markets are complete, so that the agents can trade contingent claims and achieve optimal risk sharing. Through the affine heterogeneous beliefs framework, our model can capture very general forms of disagreements among the agents while maintaining the tractability. For example, the agents can disagree about the intensity of disasters or the severity of disasters, and the amount of disagreements can fluctuate over time.

One of our main findings is that having a second type of agents with different beliefs about disasters can cause the equity premium to drop substantially, even when the new agents only have a small amount of wealth. This result holds whether the disagreement is about the intensity or impact of disasters. In fact, the result can still be true even when the new agents are generally more pessimistic about disasters. We analytically characterize the sensitivity of risk premium to the wealth distribution and derive its limit as the amount of disagreement increases. When we calibrate the beliefs of one agent using international data (from Barro (2006)) and the other using only consumption data from the US (where disasters have been relatively mild), raising the fraction of total wealth for the second agent from 0 to 10% lowers the equity premium from 4.4% to 2.0%. The decline in the equity premium becomes faster when the disagreement is larger, or when the new agents also have lower risk aversion.

There are two key reasons behind this result: (1) the equity premium is highly sensitive to changes in the size of individual consumption losses during a disaster; (2) the equity premium derives almost entirely from jump (disaster) risk, which implies high prices for jump risk and induces aggressive risk sharing.

First, there is a highly nonlinear relationship between risk premium and disaster risk exposure. For example, if an agent (with $\gamma = 4$) manages to reduce her consumption loss in a disaster from 40% to 35%, the equity premium she demands will fall by 40%! This non-linearity is an intrinsic property of disaster models, which generate high premium from rare events by making marginal utility in the disaster states rise substantially with the size of the consumption losses. As a result, a small reduction in the individual disaster risk exposure due to risk sharing can significantly lower the premium.

Second, in our economy, as is typical in standard power utility models, there is very little compensation for Brownian risk due to the low volatility of consumption and moderate levels of risk aversion. Consequently, the equity premium derives primarily from disaster risk, and the compensation for bearing disaster risk must be high. For example, if the equity premium due to disaster risk is 4%, and there is a single type of disaster resulting in a 40% loss to the market, then the annual premium for a disaster insurance contract that pays \$1 when disaster strikes must cost 10 cents or more, regardless of the actual chance of payoff.

Such a high premium for disaster risk provides strong motivation for risk sharing when agents have different beliefs about disasters. In a benchmark example of our model, the pessimists are willing to pay up to 13 cents per \$1 of disaster insurance coverage, even though the payoff probability is only 1.7% under their own beliefs. The optimists, who believe the payoff probability is just 0.1%, underwrite insurance contracts with notional value up to 40% of their total wealth, despite the risk of losing 70% of their consumption if a disaster strikes.

Taken together, when we allocate a small amount of wealth to agents with heterogeneous beliefs, the risk sharing they provide will be enough to significantly reduce the equity premium in equilibrium. Importantly, the above mechanism does not require the new agents to be "globally" more optimistic about disasters than the existing ones. What is critical to the risk sharing mechanism is that the minority wealth holders believe that the types of disasters the majority wealth holders fear most are relatively unlikely. Although these minority wealth holders may fear other disasters (perhaps even larger and/or more frequent ones), they will still be willing to share the disaster risk that the majority wealth holders fear. Thus, heterogeneity among agents may result in a low equity premium even if each would individually demand a high equity premium when other types of agents are not present.

The model not only demonstrates the sensitivity of disaster risk premium to heterogeneous beliefs, but also highlights the conditions under which disaster risk premium will be large, namely when disagreement across agents is small, or when the wealth distribution is highly concentrated in those agents with similar fears of disasters. When the wealth distribution across agents with different beliefs is not too concentrated, the disaster risk premium will remain low and smooth as the average belief of disaster risk in the market fluctuates. However, when a disaster strikes, those optimists will lose a large fraction of their wealth and their risk sharing capacity will be greatly reduced. As a result, the disaster risk premium will jump up significantly, and become more sensitive to fluctuations in disaster risk going forward. Similarly, the amount of disagreement across agents also has important effects on disaster risk. If agents' beliefs converge when disaster risk rises, that could amplify the rise of the disaster risk premium. However, if beliefs diverge, the disaster risk premium can actually become lower just as the average perceived disaster risk rises.

A number of other interesting results and predictions arises from our analysis. We show that agents who are overly optimistic about disasters are likely to survive and even gain wealth for long periods of time. This is quite different from the case of disagreement about mean growth rates, where agents with wrong beliefs are likely to lose the majority of their wealth quickly. Also, similar to the link between asset prices and the size of the market for riskless lending in Longstaff and Wang (2008), our model predicts an inverse U-shaped relationship between the equity premium and the size of the disaster insurance market.

This paper builds on the disaster risk model of Rietz (1988), Longstaff and Piazzesi (2004), and Barro (2006). Barro has reinvigorated this literature by providing international evidence that disasters have been frequent and severe enough to generate a large equity premium.[1] The majority of these studies adopt a representative-agent framework. The two papers closest to ours are Bates (2008) and Dieckmann (2009). Bates (2008) studies investors with heterogenous attitudes towards crash risk, which is isomorphic to heterogeneous beliefs of disaster risk. He focuses on small but frequent crashes and does not model intermediate consumption. Dieckmann considers only log utility. In these settings, risk sharing has limited effects on the equity premium. In addition, our model also captures more general disagreements about disasters, time-varying disaster intensities, and time-varying disagreement.

The paper also contributes to the literature of heterogeneous beliefs and preferences.[2] Our affine heterogeneous beliefs framework makes it tractable to study various forms of heterogeneity in beliefs about disasters through the generalized transform results of Chen and Joslin (2009). Our main finding is related to the results of Kogan, Ross, Wang, and Westerfield (2006), who show that irrational traders can still have large price impact when their wealth becomes negligible.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 analyzes the effect of risk sharing in a setting with disagreement about disaster

[1] A series of recent studies include Liu, Pan, and Wang (2005), Weitzman (2007), Barro (2009), Gabaix (2009), Wachter (2009), Martin (2008), Farhi and Gabaix (2009), Backus, Chernov, and Martin (2009), and others.

[2] See Basak (2005) for a survey on heterogeneous beliefs and asset pricing. Recent developments include Kogan, Ross, Wang, and Westerfield (2006), Buraschi and Jiltsov (2006), Yan (2008), David (2008), Dumas, Kurshev, and Uppal (2009), Xiong and Yan (2009), among others. Among the works on heterogeneous preferences are Dumas (1989), Wang (1996), Chan and Kogan (2002), and more recently Longstaff and Wang (2008).

intensity. Section 4 generalizes the forms of disagreements and calibrates two sets of beliefs using historical data. Section 5 studies the effects of time-varying disagreement. Section 6 concludes.

2 Model Setup

We consider a continuous-time, pure exchange economy. There are two agents (A, B), each being the representative of her own class. Agent A believes that the aggregate endowment is $C_t = e^{c_t^c + c_t^d}$, where c_t^c is the diffusion component of log aggregate endowment, which follows

$$dc_t^c = \bar{g}dt + \sigma_c dW_t^c, \tag{1}$$

where $\bar{g}$ and σ_c are the expected growth rate and volatility of consumption without jumps, and W_t^c is a standard Brownian motion under agent A's beliefs. The term c_t^d is a pure jump process whose jumps arrive with stochastic intensity λ_t,

$$d\lambda_t = \kappa(\bar{\lambda}^A - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dW_t^\lambda, \tag{2}$$

where $\bar{\lambda}^A$ is the long-run average jump intensity under A's beliefs, and W_t^λ is a standard Brownian motion independent of W_t^c. The jumps Δc_t^d have time-invariant distribution ν^A. We summarize agent A's beliefs with the probability measure $\mathbb{P}_A$.

Agent B believes that the probability measure is $\mathbb{P}_B$, which we shall suppose is equivalent to $\mathbb{P}_A$.[3] She may disagree about the growth rate of consumption without jumps, the likelihood of disasters or the distribution of the severity of disasters when they occur. We assume that the two agents are aware of each others' beliefs, but nonetheless "agree to disagree".[4]

[3] More precisely, $\mathbb{P}^A$ and $\mathbb{P}^B$ are equivalent when restricted to any σ-field $\mathcal{F}_T = \sigma(\{c_t^c, c_t^d, \lambda_t\}_{0\le t\le T})$.

[4] We do not explicitly model learning about disasters. Given the nature of disasters, Bayesian updating of beliefs about disaster risk using realized consumption growth will likely be very slow, and

Specifically, as in Chen, Joslin, and Tran (2010), agent B's beliefs are characterized by the Radon-Nikodym derivative $\eta_t \equiv (d\mathbb{P}_B/d\mathbb{P}_A)_t$, which satisfies

$$\eta_t = e^{a_t + bc_t^c - I_t}\,, \tag{3}$$

$$I_t = \int_0^t \left(b\bar{g} + \frac{1}{2}b^2\sigma_c^2 + \lambda_s \left(\frac{\bar{\lambda}^B}{\bar{\lambda}^A} - 1) \right) \right) ds\,, \tag{4}$$

for some constants b and $\bar{\lambda}^B > 0$, and a_t is a pure jump process whose jumps are coincident with the jumps in c_t^d and have size

$$\Delta a_t = \log\left(\frac{\bar{\lambda}^B}{\bar{\lambda}^A} \frac{d\nu^B}{d\nu^A} \right), \tag{5}$$

where $\frac{d\nu^B}{d\nu^A}$ is a function of the disaster size and reflects the disagreement about the distribution of disaster size (conditional on a disaster). It will be large (small) for the type of disasters that agent B thinks are relatively more (less) likely than agent A.

Intuitively, η_t expresses the differences in beliefs between the agents by letting agent B assign a higher probability to those states where η_t is large. The terms a_t and bc_t^c reflect B's potential disagreements regarding the likelihood of disasters and the growth rate of consumption, respectively. It follows from (3-5) that, under agent B's beliefs, the expected growth rate of consumption without jumps is $\bar{g} + b\sigma_c^2$, a disaster occurs with intensity $\lambda_t \times \frac{\bar{\lambda}^B}{\bar{\lambda}^A}$ (with long run average intensity $\bar{\lambda}^B$), and the disaster size distribution is ν^B (which is equivalent to ν^A). The jumps in η_t specified in (5) are given by the log likelihood ratio for disasters of different sizes under the two agents' beliefs. Within this setup, agent B not only can disagree with A on the average frequency of disasters, but also the likelihoods for disasters of different magnitude. Moreover, this setup also has the advantage of remaining within the affine family as $(c_t^c, c_t^d, \log \eta_t, \lambda_t)$ follows a jointly affine process, which makes it possible to compute the equilibrium analytically.

We assume that the agents are infinitely lived and have constant relative-risk aver-

the disagreements in the priors will persist for a long time.

sion (CRRA) utility over life time consumption:

$$U^i(C^i) = E_0^{\mathbb{P}_i}\left[\int_0^\infty e^{-\rho_i t}\frac{(C_t^i)^{1-\gamma_i}}{1-\gamma_i}dt\right], \quad i = A, B. \tag{6}$$

We also assume that markets are complete and agents are endowed with some fixed share of aggregate consumption ($\theta_A, \theta_B = 1 - \theta_A$).

The equilibrium allocations can be characterized as the solution of the following planner's problem, specified under the probability measure $\mathbb{P}_A$,

$$\max_{C_t^A, C_t^B} E_0^{\mathbb{P}_A}\left[\int_0^\infty e^{-\rho_A t}\frac{(C_t^A)^{1-\gamma_A}}{1-\gamma_A} + \tilde{\zeta}_t e^{-\rho_B t}\frac{(C_t^B)^{1-\gamma_B}}{1-\gamma_B}dt\right], \tag{7}$$

subject to the resource constraint $C_t^A + C_t^B = C_t$. Here, $\tilde{\zeta}_t \equiv \zeta\eta_t$ is the belief-adjusted Pareto weight for agent B. From the first order condition and the resource constraint we obtain the equilibrium consumption allocations: $C_t^A = f^A(\hat{\zeta}_t)C_t$ and $C_t^B = (1 - f^A(\hat{\zeta}_t))C_t$, where $\hat{\zeta}_t = e^{(\rho_A-\rho_B)t}C_t^{\gamma_A-\gamma_B}\tilde{\zeta}_t$, and f^A is in general an implicit function.

The stochastic discount factor under A's beliefs, M_t^A, is given by

$$M_t^A = e^{-\rho_A t}(C_t^A)^{-\gamma_A} = e^{-\rho_A t}f^A(\hat{\zeta}_t)^{-\gamma_A}C_t^{-\gamma_A}. \tag{8}$$

Finally, we can solve for ζ through the life-time budget constraint for one of the agents (see Cox and Huang (1989)), which is linked to the initial allocation of endowment.

Since our emphasis is on heterogeneous beliefs about disasters, for the remainder of this section we focus on the case where there is no disagreement about the distribution of Brownian shocks, and the two agents have the same preferences. In this case, $b = 0$, $\gamma_A = \gamma_B = \gamma$, $\rho_A = \rho_B = \rho$. The equilibrium consumption share then simplifies to

$$f^A(\tilde{\zeta}_t) = \frac{1}{1 + \tilde{\zeta}_t^{\frac{1}{\gamma}}}\ . \tag{9}$$

When a disaster of size d occurs, $\tilde{\zeta}_t$ is multiplied by the likelihood ratio $\frac{\bar{\lambda}^B}{\bar{\lambda}^A}\frac{d\nu^B}{d\nu^A}(d)$ (see

(5)). Thus, if agent B is more pessimistic about a particular type of disaster, she will have a higher weight in the planner's problem when such a disaster occurs, so that her consumption share increases.

The equilibrium allocations can be implemented through competitive trading in a sequential-trade economy. Extending the analysis of Bates (2008), we can consider three types of traded securities: (i) a risk-free money market account, (ii) a claim to aggregate consumption, and (iii) a series (or continuum) of disaster insurance contracts with 1 year maturity, which pay \$1 on the maturity date if a disaster of size d occurs within a year.

The instantaneous risk-free rate can be derived from the stochastic discount factor,

$$r_t = -\frac{\mathcal{D}^A M_t^A}{M_t^A} = \rho + \gamma\bar{g} - \frac{1}{2}\gamma^2\sigma_c^2 - \lambda_t\left(\frac{E_t^{\Delta,A}[(C_t^A)^{-\gamma}]}{(C_t^A)^{-\gamma}} - 1\right), \tag{10}$$

where $\mathcal{D}^A$ denotes the infinitesimal generator under Agent A's beliefs of $X_t = (c_t^c, c_t^d, \lambda_t, \eta_t)$ and we use the short-hand notation $E_t^{\Delta,A}$ defined for any function f of X_t as

$$E_t^{\Delta,A}[f(X_t)] \equiv \int f\left(c_t^c, c_t^d + d, \lambda_t, \eta_t \times \frac{\bar{\lambda}^B}{\bar{\lambda}^A}\frac{d\nu^B}{d\nu^A}(d)\right) d\nu^A(d). \tag{11}$$

The price of the aggregate endowment claim is

$$P_t = \int_0^\infty E_t^{\mathbb{P}_A}\left[\frac{M_{t+\tau}^A}{M_t^A}C_{t+\tau}\right] d\tau \ = C_t h(\lambda_t, \tilde{\zeta}_t)\,, \tag{12}$$

where the price/consumption ratio only depends on the disaster intensity λ_t and the stochastic weight $\tilde{\zeta}_t$. In the case where λ_t is constant, the price of the consumption claim is obtained in closed form. Similarly, we can compute the wealth of the individual agents as well as the prices of disaster insurance contracts using the stochastic discount factor (see Appendix A for details).

In order for prices of the aggregate endowment claim to be finite in the heterogeneous-agent economy, it is necessary and sufficient that prices are finite under each agent's beliefs in a single-agent economy (see Appendix C for a proof). As we show in the appendix, finite prices require that the following two inequalities hold:

$$0 < \kappa^2 - 2\sigma_\lambda^2(\phi^i(1-\gamma)-1), \tag{13a}$$

$$0 > \kappa\bar{\lambda}^i \frac{\kappa - \sqrt{\kappa^2 + 2\sigma_\lambda^2(1-\phi^i(1-\gamma))}}{\sigma_\lambda^2} - \rho + (1-\gamma)\bar{g} + \frac{1}{2}(1-\gamma)^2\sigma_c^2, \tag{13b}$$

where ϕ^i is the moment generating function for the distribution of jumps in endowment ν^i under measure $\mathbb{P}_i$. The first inequality reflects the fact that the volatility of the disaster intensity cannot be too large relative to the rate of mean reversion. It prevents the convexity effect induced by the potentially large intensity from dominating the discounting. The second inequality reflects the need for enough discounting to counteract the growth.

Additionally, the stochastic discount factor characterizes the unique risk neutral probability measure $\mathbb{Q}$ (see, for example, Duffie (2001)), which facilitates the computation and interpretation of excess returns. The risk-neutral disaster intensity $\lambda_t^{\mathbb{Q}} \equiv E_t^{\Delta,i}[M_t^i]/M_t^i\lambda_t^i$ is determined by the expected jump size of the stochastic discount factor at the time of a disaster. When the riskfree rate and disaster intensity are close to zero, the risk-neutral disaster intensity $\lambda_t^{\mathbb{Q}}$ has the nice interpretation of (approximately) the value of a one-year disaster insurance contract that pays \$1 at $t+1$ when a disaster occurs between t and $t+1$. The risk-neutral distribution of the disaster size is given by $\frac{d\nu^{\mathbb{Q}}}{d\nu^i}(d) = M_t^{i,\Delta}(d)/E_t^{\Delta,i}[M_t^i]$, where $M_t^{i,\Delta}(d)$ denotes the pricing kernel when the state is $(c_t^c, c_t^d + d, \lambda_t, \eta_t \times \frac{\bar{\lambda}^B}{\bar{\lambda}^A}\frac{d\nu^B}{d\nu^A}(d))$. These risk adjustments are quite intuitive. The more the stochastic discount factor for agent i jumps up during a disaster, the large is $\lambda_t^{\mathbb{Q}}$ relative to λ_t^i, i.e. disasters occur more frequently under the risk-neutral measure. Thus, the ratio $\lambda_t^{\mathbb{Q}}/\lambda_t^i$ is often referred to as the jump-risk premium. Moreover, the risk-adjusted distribution of jump size conditional on a disaster slants the probabilities towards the types of disasters that lead to a bigger jump in the stochastic discount factor, which generally makes severe disasters more likely under $\mathbb{Q}$.

Finally, the risk premium for any security under agent i's beliefs is the difference between the expected return under $\mathbb{P}_i$ and under the risk-neutral measure $\mathbb{Q}$. In the case of the aggregate endowment claim, the conditional equity premium, under agent i's beliefs,which we denote by $E_t^{\mathbb{P}_i}[R^e]$, is

$$E_t^{\mathbb{P}_i}[R^e] = \gamma\sigma_c^2 + \lambda_t^i E_t^{\mathbb{P}_i}[\Delta R] - \lambda_t^{\mathbb{Q}} E_t^{\mathbb{Q}}[\Delta R], \quad i = A, B \tag{14}$$

where $E_t^m[\Delta R] \equiv E_t^{\Delta,m}[P_t]/P_t - 1$ is the expected return on the endowment claim in a disaster under measure m.[5] The difference between the last two terms in (14) is the premium for bearing disaster risk. This premium is large if the jump-risk premium is large, and/or the expected loss in return in a disaster is large (especially under the risk-neutral measure).

It follows that the difference in equity premium under the two agents' beliefs is

$$E_t^{\mathbb{P}_A}[R^e] - E_t^{\mathbb{P}_B}[R^e] = \lambda_t^A E_t^{\mathbb{P}_A}[\Delta R] - \lambda_t^B E_t^{\mathbb{P}_B}[\Delta R]\,.$$

This difference will be small relative to the size of the equity premium when the disaster intensity and expected loss under the risk-neutral measure are large relative to their values under actual beliefs. In the remainder of the paper, we report the equity premium relative to agent A's beliefs, $\mathbb{P}_A$. One interpretation for picking $\mathbb{P}_A$ as the reference measure is that A has the correct beliefs, and we are studying the impact of the incorrect beliefs of agent B on asset prices.

3 Heterogeneous Beliefs and Risk Sharing

We start with a special case of the model where agents only disagree about the frequency of disasters. First, we analyze the impact of heterogeneous beliefs and its

[5] To be concrete, we define the risk premium under measure i for any price process $P(X_t, t)$ which pays dividends $D(X_t, t)$ to be $\mathcal{D}^i P/P - (r_t + D_t)$.

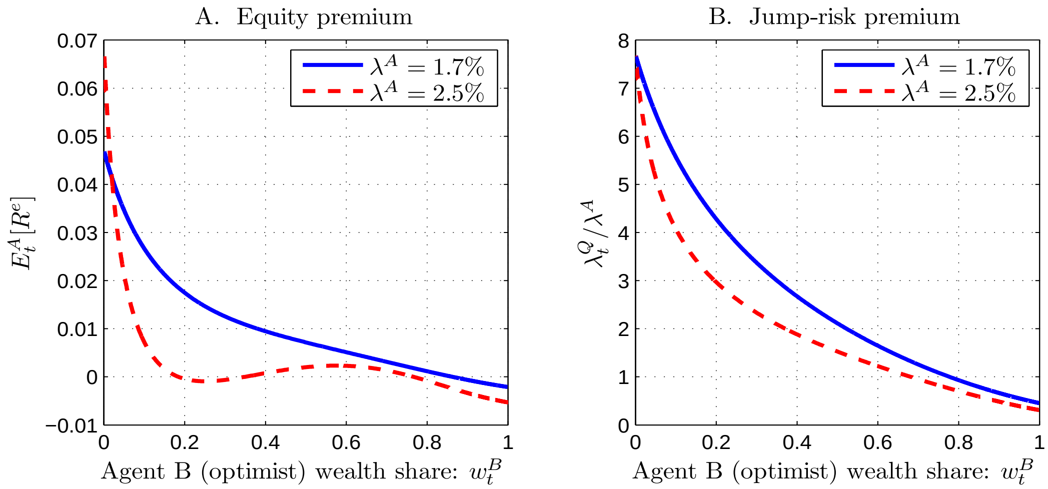

Figure 1: **Disagreement about the frequency of disasters.** Panel A plots the equity premium under the pessimist's beliefs as a function of the wealth share of the optimist. Panel B plots the jump-risk premium $\lambda_t^{\mathbb{Q}}/\lambda^A$ for the pessimist.

implications for survival when the risk of disasters is constant, i.e., $\lambda_t = \bar{\lambda}^A$ (denoted as λ^A for simplicity). We then extend the analysis to the case with time-varying disaster risk.

3.1 Disagreement about the Frequency of Disasters

In the simplest version of our model, the disaster size is deterministic, $\Delta c_t^d = \bar{d}$, and the two agents only disagree about the frequency of disasters (λ). We set $\bar{d} = -0.51$ so that the moment generating function (MGF) $\phi^A(-\gamma)$ in this model matches the calibration of Barro (2006) for $\gamma = 4$. It implies that aggregate consumption falls by 40% when a disaster occurs. Agent A (pessimist) believes that disasters occur with intensity $\lambda^A = 1.7\%$ (once every 60 years), which is also taken from Barro (2006). Agent B (optimist) believes that disasters are much less likely, $\lambda^B = 0.1\%$ (once every 1000 years), but she agrees with A on the size of disasters as well as the Brownian risk in consumption. She also has the same preferences as agent A. The remaining parameters are $\bar{g} = 2.5\%$, $\sigma_c = 2\%$, and $\rho = 3\%$.

Figure 1 shows the conditional equity premium and the jump-risk premium under the pessimist's beliefs. If all the wealth is owned by the pessimist, the equity premium is 4.7%, and the riskfree rate is 1.3%. Since the optimist assigns very low probabilities to disasters, if she has all the wealth, the equity premium is only -0.21% under the pessimist's beliefs, which reflects the low compensation the optimist requires for bearing disaster risk and the higher frequency of the pessimist beliefs. Thus, it is not surprising to see the premium falling when the optimist owns more wealth. However, the speed at which the premium declines in Panel A is impressive. When the optimistic agent owns 10% of the total wealth, the equity premium has fallen from 4.7% to 2.7%. When the wealth of the optimist reaches 20%, the equity premium falls to just 1.7%.

We can derive the conditional equity premium as a special case of (14), where the assumption of constant disaster size helps simplify the expression:

$$E_t^{\mathbb{P}_A}[R^e] = \gamma\sigma_c^2 - \lambda^A\left(\frac{\lambda_t^{\mathbb{Q}}}{\lambda^A} - 1\right)\left(\frac{h(\tilde{\zeta}_t\frac{\lambda^B}{\lambda^A})e^{\bar{d}}}{h(\tilde{\zeta}_t)} - 1\right), \tag{15}$$

where h is the price-consumption ratio from (12), with λ_t being constant. The first term $\gamma\sigma_c^2$ is the standard compensation for bearing Brownian risk. Heterogeneity has no effect on this term since the agents agree about the brownian risk. Given the value of risk aversion and consumption volatility, this term has negligible effect on the premium. The second term reflects the compensation for disaster risk. It can be further decomposed into three factors: (i) the constant disaster intensity λ^A, (ii) the jump-risk premium $\lambda_t^{\mathbb{Q}}/\lambda^A$, and (iii) the return of the consumption claim in a disaster.

How does the wealth distribution affect the jump-risk premium? From the definition of the stochastic discount factor M_t^A and the risk-neutral intensity $\lambda_t^{\mathbb{Q}}$, it is easy to show $\lambda_t^{\mathbb{Q}}/\lambda^A = e^{-\gamma\Delta c_t^A}$, where Δc_t^A is the jump size of the equilibrium log consumption for agent A in a disaster, which could be very different from the jump size in aggregate endowment due to trading. Without trading $\Delta c_t^A = \bar{d}$, which generates a jump-risk premium of $\lambda_t^{\mathbb{Q}}/\lambda^A = 7.7$. Since $\lambda_t^{\mathbb{Q}}$ is approximately the premium of a one-year disaster insurance, before any trading the pessimist will be willing to pay an annual premium of about 13 cents for \$1 of protection against a disaster event that occurs with probability 1.7%.

Since the optimist views disasters as very unlikely events, she is willing to trade away her claims in the future disaster states in exchange for higher consumption in normal times. For example, she will find selling an \$1 disaster insurance and collecting a 13 cents premium a lucrative trade. Such a trade helps reduce the pessimist's consumption loss in a disaster Δc_t^A, which in turn lowers the jump-risk premium. However, the optimist's capacity for underwriting disaster insurance is limited by her wealth, as she needs to ensure that her consumption/wealth is positive in all future states, including when a disaster occurs (no matter how unlikely such an event is). Thus, the more wealth the optimist has, the more disaster insurance she is able to sell without making her consumption too risky when a disaster strikes.

The above mechanism can substantially reduce the disaster risk exposure of the pessimist in equilibrium. Panel B of Figure 1 shows that the jump-risk premium falls rapidly. When the optimist owns 20% of total wealth, the jump-risk premium drops from 7.7 to 4.2. According to equation (15), such a drop in the jump-risk premium alone will cause the equity premium to fall by about half to 2.2%, which accounts for the majority of the change in the premium (from 4.7% to 1.7%).

Besides the jump-risk premium, the equity premium also depends on the return of the consumption claim in a disaster, which in turn is determined by the consumption loss and changes in the price-consumption ratio. Following a disaster, the riskfree rate drops as the wealth share of the pessimist rises. With CRRA utility, the lower interest rate effect can dominate that of the rise in the risk premium, leading to a higher price-consumption ratio.[6] Since a higher price-consumption ratio partially offsets the drop in aggregate consumption, it makes the return less sensitive to disasters, which will

[6]Wachter (2009) also finds a positive relation between the price-consumption ratio and the equity premium in a representative agent rare disaster model with time-varying disaster probabilities and CRRA utility.

contribute to the drop in equity premium. However, our decomposition above shows that the reduction of the jump-risk premium (due to reduced disaster risk exposure) is the main reason behind the fall in premium.

Can we "counteract" the effect of the optimistic agent and restore the high equity premium by making the pessimist even more pessimistic about disasters? The dashed lines in Figure 1 plot the results when agent A believes that the disaster intensity is 2.5% ($\lambda^A = 2.5\%$) and everything else equal. The results are striking. While the equity premium becomes significantly higher (6.8%) when the pessimist owns all the wealth, it falls to 4.1% with just 2% of total wealth allocated to the optimist (already lower than the previous case with $\lambda^A = 1.7\%$), and is below 1% when the wealth of the optimist exceeds 8.5%. As the wealth share of the optimist grows higher, the premium can even become negative. The decline in the jump-risk premium is still the main reason behind the lower equity premium. For example, when the optimist has 10% of total wealth, the jump-risk premium falls to 4.0, which will drive the premium down to 3.1% (60% of the total fall). Thus, as the pessimist becomes more pessimistic, she seeks risk sharing more aggressively, which can quickly reverse the effect of her heightened fear of disasters.

To better illustrate the risk sharing mechanism between agents, we compute their portfolio positions in the aggregate consumption claim, disaster insurance, and the money market account. Calculating these portfolio positions amounts to finding a replicating portfolio that matches the exposure to Brownian shocks and jumps in the individual agents' wealth processes. Appendix B provides the details. The first thing to notice is that each agent will hold a constant proportion of the consumption claim. This is because they agree on the brownian risk and share it proportionally. Disagreement over disaster risk is resolved through trading in the disaster insurance market, which is financed by the money market account.

We first plot the notional value of the disaster insurance sold by the optimist as a fraction of her total wealth in Panel A of Figure 2. The dashed line is the maximum

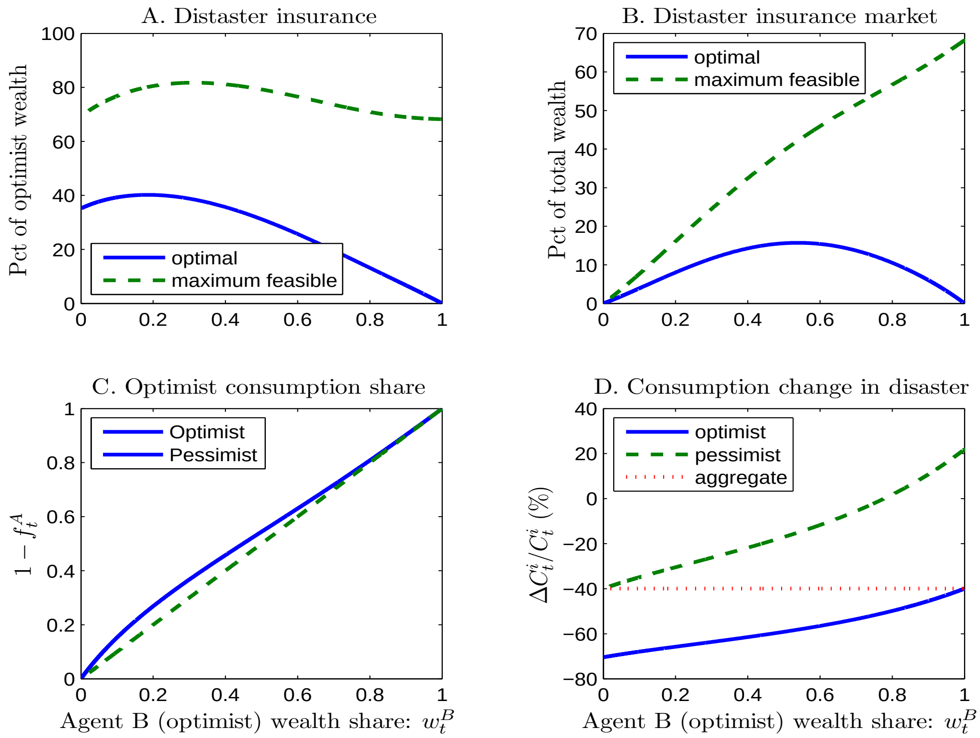

Figure 2: **Risk sharing.** Panel A and B plot the total notional value of disaster insurance relative to the wealth of the optimist and total wealth in the economy. Panel C plots the consumption share for the optimist in equilibrium. Panel D compares the two agents' consumption drops in a disaster with that of the aggregate endowment. These results are for the case $\lambda^A = 1.7\%$.

amount of disaster insurance the optimist can sell (as a fraction of her wealth) subject to her budget constraint. When the optimist has very little wealth, the notional value of the disaster insurance she sells is about 35% of her wealth. This value initially rises and then falls as the optimist gains more wealth. When the optimist has little wealth, the pessimist has great demand for risk sharing and is willing to pay a higher premium, which induces the optimist to sell more insurance relative to her wealth. As the optimist gets more wealth, the premium on the disaster insurance falls, and so does the relative amount of insurance sold.

We can judge how extreme the risk sharing in equilibrium is by comparing the actual amount of trading to its limit. At its peak, the amount of disaster insurance sold by the optimist is about half of the maximum amount that she can underwrite, which might appear reasonable. The caveat is that, in reality, underwriters of disaster insurance will likely be required to collateralize their promises to pay in the disaster states, which raises the costs of risk sharing.[7]

Panel B plots the size of the disaster insurance market (the total notional value normalized by total wealth). Naturally, the size of this market is zero when either agent has all the wealth, and the market is bigger when wealth is more evenly distributed. Notice that the model generates a non-monotonic relation between the size of the disaster insurance market and the equity premium. The premium is high when there is a lot of demand for disaster insurance but little supply, and is low when the opposite is true. In either case, the size of the disaster insurance market will be small.

Panel C plots the equilibrium consumption share for the optimist. The 45-degree line corresponds to the case of no trading. The optimist's consumption share is above the 45-degree line, especially when her wealth is small. This is because the optimist is giving up consumption when disasters occur in order to have more consumption now (and in the future, provided a disaster has not occurred.) Panel D shows that indeed the optimist does bear much greater losses in the event of a disaster in order to sustain higher current consumption. As for the pessimist, the less wealth she possesses, the more disaster insurance she buys relative to her wealth. This will gradually lower her disaster risk exposure, and can eventually turn the disaster insurance into a speculative position — her consumption can jump up in a disaster. Finally, if we make agent A's beliefs more pessimistic (e.g. $\lambda^A = 2.5\%$), the amount of disaster insurance traded (both relative to the wealth of the optimistic agent and to total wealth in the economy) will become higher, while the consumption shares will become more nonlinear. As a result, the risk premium declines more rapidly with the optimist's wealth share.

[7]The collateral constraint can be especially important when agents' wealth is mostly in the form of future labor income.

3.2 The Limiting Case for Risk Sharing

In order to highlight the key ingredients of the risk sharing mechanism demonstrated in the previous section, we now characterize the properties of the equilibrium when a small fraction of the wealth is controlled by an optimist who believe disasters are extremely unlikely.[8] Consider the effect of a disaster at time t on the marginal utility of the pessimistic agent (agent A). Before the disaster, suppose that the equilibrium consumption of the optimist agent is a fraction f^A_{t-} of the aggregate endowment C_{t-}. After the disaster, the aggregate endowment drops to $C_t = e^{\bar{d}} C_{t-}$ but now the pessimistic agent consumes essentially the entire endowment (i.e. $f^A_t \approx 1$). This is because the optimist feels disasters are so unlikely that she is willing to sell all her share of the endowment in this state to the pessimist. Such a jump in marginal utility is associated with a jump risk premium of

$$\frac{\lambda^{\mathbb{Q}}_{t-}}{\lambda^A} \approx \frac{\left(1 \times e^{\bar{d}} C_{t-}\right)^{-\gamma}}{\left(f^A_{t-} C_{t-}\right)^{-\gamma}} = \left(f^A_{t-}\right)^{\gamma} e^{-\gamma \bar{d}}. \tag{16}$$

For example, when the optimist has only 1% of the endowment to give up in disasters to the pessimist, this decreases the jump risk premium from $e^{-\gamma \bar{d}}$ to $(.99)^{\gamma} e^{-\gamma \bar{d}}$, or approximately a 4% drop when $\gamma = 4$.

Formally, we show in the Appendix D that

$$\lim_{\lambda^B \to 0^+} \frac{\partial}{\partial f^B_t} \frac{\lambda^{\mathbb{Q}}_t}{\lambda^A}\bigg|_{f^B_t = 0} = -\gamma e^{-\gamma \bar{d}}. \tag{17}$$

Thus we see that the effect of risk sharing (in terms of consumption share) becomes stronger (i) when the size of the disaster increases and (ii) when risk aversion increases.[9]

This only partially reflects the steep slope in the risk premium near $w^B_t = 0$ we see

[8]We thank Xavier Gabaix for suggesting this analysis.

[9]We take limits since with $\lambda^B = 0$, the beliefs are not equivalent and multiple equilibria are possible.

in Figure 1. Also reflected is the fact that if the optimist consumes a fraction f_t^B of the endowment at time t, his fraction of the aggregate wealth, w_t^B, must be *less* than f_t^B. This is because the optimist consumes nothing in the valuable disaster states because she thinks they are unlikely to occur. We can approximate this relation as follows. First consider a claim on the aggregate endowment from now to the time of the first disaster. The value of such a claim at a time when the pessimist has all the wealth is

$$V_{ND} \equiv \int_0^\infty E_0 \left[\frac{M_t^A}{M_0^A} C_t \times 1_{\{N_t=0\}} \middle| f_0^A = 1 \right] dt = \frac{1}{\rho + (\gamma-1)\bar{g} - \frac{1}{2}\sigma_c^2(1-\gamma)^2 + \lambda^A}. \tag{18}$$

Next, the value of the entire endowment claim is

$$V \equiv \int_0^\infty E_0 \left[\frac{M_t^A}{M_0^A} C_t \middle| f_0^A = 1 \right] dt = \frac{1}{\rho + (\gamma-1)\bar{g} - \frac{1}{2}\sigma_c^2(1-\gamma)^2 - \lambda^A(e^{(1-\gamma)\bar{d}} - 1)}. \tag{19}$$

When the optimist current equilibrium consumption fraction of the entire endowment is f_t^B (close to 0), his wealth fraction is then approximately $f_t^B V_{ND}/V$. For example, in the calibration of Section 4.1, the price-consumption ratio for the claim to the entire endowment when $f_t^A = 1$ is 23.98 while the price-consumption ratio of the claim to the endowment until the first disaster is 8.32. Using the approximation, it will be that in the limiting case of $\lambda^B = 0$ we have $f^B/w^B \approx 23.98/8.32 = 2.88$. Thus, if the optimist has 1% of the aggregate wealth, he chooses to consume 2.88% of the endowment until a disaster occurs, at which point he consumes nothing (approximately.)

In Appendix D, we show formally that

$$\lim_{\lambda^B \to 0^+} \left. \frac{\partial f_t^B}{\partial w_t^B} \right|_{f_t^B = 0} = \frac{\rho + (\gamma-1)\bar{g} - \frac{1}{2}\sigma_c^2(1-\gamma)^2 + \frac{\gamma-1}{\gamma}\lambda^A}{\rho + (\gamma-1)\bar{g} - \frac{1}{2}\sigma_c^2(1-\gamma)^2 - \lambda^A(e^{(1-\gamma)\bar{d}} - 1)}. \tag{20}$$

This numerator in the limiting expression differs slightly from (18) in that it reflects the fact that optimist will gain weatlh throughout time when no disaster occurs from selling disaster insurance to the pessimist. This effect will be larger when disasters are more frequent and larger. Additionally, when (i) the effect of disasters dominate the effect of growth in the sense that $\lambda^A e^{(1-\gamma)\bar{d}} > (\gamma - 1)\bar{g}$ and (ii) volatility induced convexity effects are small, increasing risk aversion will increase this multiplier.

In the calibrated example, the more precise limiting expression in (20) gives a consumption-wealth multiplier of 2.78.[10] Numerically, we compute the (non-limiting value) when $\lambda^B = 0.1\%$ as a consumption-wealth multiplier of 1.86. Combining these effects we see that the limiting derivative with respect to wealth fraction as λ^B approaches zero of the jump risk premium is -85.5, or -11.1 times the maximum jump risk premium. Thus allocating 1% of the wealth to extreme optimist reduce the jump risk premium by 11.1% of the maximum value in the limit.

We can summarize the effect of the disaster risk premium on the equity premium from the decomposition in (14). Ignoring the effects on the disaster return generated by risk sharing[11], the limiting differential effect of optimist on risk premia is given by

$$\lim_{\lambda^B \to 0^+} -(e^{\bar{d}} - 1)\frac{\partial \lambda_t^{\mathbb{Q}}/\lambda^A}{\partial w_t^B}\bigg|_{f_t^B=0} = -\frac{\partial \lambda_t^{\mathbb{Q}}/\lambda^A}{\partial f_t^B}\bigg|_{f_t^B=0} \times \frac{\partial f_t^B}{\partial w_t^B}\bigg|_{f_t^B=0} \times (e^{\bar{d}} - 1) \times \lambda^A. \quad (21)$$

In the calibrated example, the limiting factor (with $\lambda^B = 0$) equals -0.581. So allocating only 1% of the endowment to extreme optimist results in a decline of 58.1 basis points in the equity premium due to this effect. The value of $\lambda^B = 0.1\%$ results in a factor of -0.19, indicating that the fact that these agents sell most, but not all, of these claims to disaster states attenuates the effect to a fair degree.[12]

Figure 3 compares the jump risk premium for several cases. First, the dotted line denotes the benchmark case from Section 3.1. We also plot the jump risk premium with the same parameters but for the limiting case where λ^B approaches zero. Additionally,

[10]The relatively small fraction of value associated with non-disaster states (1/2.78) may be surprising, given the low likelihood of the disaster. In Appendix E we show, in fact, that this is a rather robust feature of models with alternative preferences so long as they feature a significant disaster component to the equity premium and a moderate wealth-consumption ratio.

[11]This is due to the fact that in disasters pessimist accumulate wealth (relatively) which drives up the price-consumption ratio due to their savings motives which partially offsets the decrease in consumption associated with the disasters.

[12]In the more severe calibration with $\lambda^A = 2.5\%$, the limiting factor is 294.3 resulting in an 2.94% drop in the equity premium by introducing only 1% of extreme optimist into the economy!

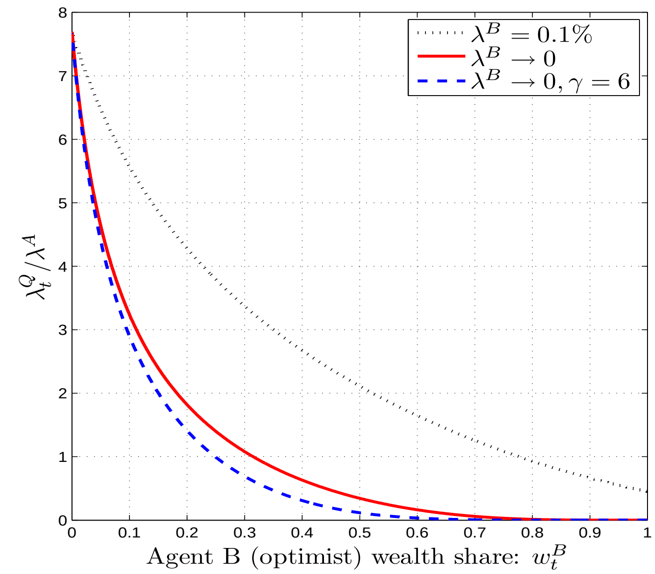

Figure 3: **Limiting Jump Risk Premia.** This figure plots the jump-risk premium $\lambda_t^{\mathbb{Q}}/\lambda^A$ for the pessimist, where $\lambda^A = 1.7\%$.

we plot the case where we decrease the disaster size and increase the risk aversion to maintain the same jump risk premium for the single agent economy ($\gamma = 5, \bar{d} = 0.408$). First, we see that marginal effect of adding a small amount of optimist with $\lambda^B = 0.1\%$ are less severe than the limiting case of extreme optimism. Second, when we decrease the size of the disaster, but increase risk aversion, the effects become more severe. This is because the risk sharing effect given in (17) increase and dominate the small reduction in consumption-wealth effect given in (20).

3.3 Survival

In models with heterogeneous agents, one type of agents often dominates in the long-run (a notable exception is Chan and Kogan (2002); see also Borovička (2010)). Our model also has the property that the agent with correct beliefs will dominate in the long run. For example, let's assume that agent A has the correct beliefs. It is easy to verify by the strong law of large numbers that $\log \tilde{\zeta}_t \to -\infty$ almost surely. This

Table 1: **Survival of Agents who Disagree about the Frequency of Disasters.** This table provides the redistribution of wealth across a 50 year horizon in the model of Section 3.1. Future relative wealth only depends on the initial wealth, the time horizon, and the number of disasters that occur. The top panel provides the possible wealth redistributions throughout time. The bottom panel provides the probability (under each agent's beliefs) for different numbers of disasters occurring.

	Final Wealth of B after N_d Disasters			
Initial Wealth of B	$N_d = 0$	$N_d = 1$	$N_d = 2$	$N_d = 3$
1.0%	1.2%	0.6%	0.3%	0.1%
5.0%	6.1%	3.0%	1.5%	0.7%
10.0%	12.2%	6.0%	2.9%	1.4%
50.0%	55.7%	35.5%	19.3%	9.6%
99.0%	99.2%	98.3%	96.7%	93.5%
Probability under $\mathbb{P}^A$	42.7%	36.3%	15.4%	4.4%
Probability under $\mathbb{P}^B$	95.1%	4.8%	0.1%	0.0%

implies that agent A will take over the economy with probability one. We now show that although agents with incorrect beliefs about disasters may not have permanent effects on asset prices, their effects may be long-lived in the sense that these agents can retain, and even build, wealth over long horizons.

With disaster intensity, λ_t, being constant, we need only consider the distribution of the stochastic Pareto weight, $\tilde{\zeta}_t$, to analyze the wealth distribution over time. From (3), we see that $\tilde{\zeta}_t$ has a stochastic component, whereby the Pareto weight (and thus wealth) of the pessimistic agent will jump up when a disaster occurs. This is because the pessimist receives insurance payments from the optimist in a disaster. However, regardless of the occurrence of disasters, there is also a deterministic component in $\tilde{\zeta}_t$, whereby the optimist has a deterministic weight increase (and thus her relative wealth increases) which comes from collecting the disaster insurance premium. Thus, even when the pessimist has *correct* beliefs, her relative wealth will *decrease* outside of disasters. Since disasters are rare, it will be common to have extended periods without disasters, during which time an optimistic agent will gain relative wealth.

Table 1 presents a summary of the conditional distribution of wealth after 50 years for various initial wealth distributions. We report the results under the assumption that either the pessimist or the optimist has correct beliefs. If the number of disasters is either 0 or 1, the wealth of the agents remain relatively close to the original distribution. We see that the optimist is likely to retain wealth for long periods of time and will only be wiped out with the occurrence of several disasters, which is unlikely regardless of whose beliefs are correct.

The evolution of the wealth distribution over time also has important implications for the equity premium and other dynamic properties of asset prices. For example, when the initial wealth of agent B is 5% (10%), the equity premium will drop from 3.5%(2.7%) to 3.3% (2.4%) over 50 years if no disasters occurs. If after 120 years there are still no disasters, the equity premium would further drop to 2.9% (2.0%).

The survival results presented thus far stand in sharp contrast to survival in models of disagreement over Brownian consumption growth. As discussed in Section 2, it is possible to raise the equity premium under the true measure if there are agents who are pessimistic about the growth rate of consumption. For example, if the volatility of consumption is $\sigma_c = 2.0\%$, two types of agents have $\gamma = 4$ and $\rho = 3\%$, one believing (correctly) consumption growth is 2.5%, the other believing it is 0% (no disasters in either case), then the equity premium will be roughly 2.5% when the pessimist controls most of the wealth in the economy. However, even if the pessimist controls 99% of the wealth initially, her wealth share will be reduced to less than 1% after 50 years with a probability of 92.4%. Thus, even a very small amount of agents with correct beliefs will quickly dominate the economy in the Gaussian setting.

3.4 Time-varying Disaster Risk

In the previous sections we have analyzed in depth the impact of heterogeneous beliefs when disaster intensity is constant. Now we extend the analysis to allow the risk of disasters and the amount of disagreements about disasters to vary over time, which not only makes the model more realistic, but also has important implications for the dynamics of asset prices. As in Gabaix (2009) and Wachter (2009), time-varying disaster intensity serves to drive both asset prices and expected excess returns. We now demonstrate that within our framework, wealth distribution becomes an important factor that drives asset price dynamics through the risk sharing mechanism. In particular, it affects how sensitive the conditional risk premium will be to time variation in disaster risk.

Our calibration of the intensity process λ_t in equation (2) is as follows. First, the long-run mean intensity of disasters under the two agents' beliefs are $\bar{\lambda}^A = 1.7\%$ and $\bar{\lambda}^B = 0.1\%$. Next, following Wachter (2009), we set the speed of mean reversion $\kappa = 0.142$ (with a half life of 4.9 years). The volatility parameter is $\sigma_\lambda = 0.05$, so that the Feller condition is satisfied.[13] For simplicity, we assume that the size of disasters is constant, $\bar{d} = -0.51$, as in Section 3.1. The remaining preference parameters are also the same as in the constant disaster risk case.

Figure 4 plots the conditional equity premium and the jump-risk premium under agent A's beliefs as functions of agent B's wealth share w_t^B and the disaster intensity λ_t. First, in Panel A, holding λ_t fixed, the equity premium drops quickly as the wealth share of the optimistic agent rises from zero, which is consistent with the results from the case with constant disaster risk. Moreover, this decline is particularly fast when λ_t is large, suggesting that the agents engage in more risk sharing when disaster risk is high. Indeed, the jump-risk premium in Panel B also declines faster when λ_t is large, which is the result of agent A reducing her consumption loss in a disaster more aggressively at such times.

Next, we see that the sensitivity of the equity premium to disaster intensity can be very different depending on the wealth distribution. The sensitivity is largest minority wealth holders has all the wealth, but it becomes smaller as the wealth of the optimist increases. When the optimist's wealth share becomes sufficiently high, the equity

[13]The Feller condition, $2\kappa\bar{\lambda}^A > \sigma_\lambda^2$, ensures that λ_t will remain strictly positive under agent A's beliefs.

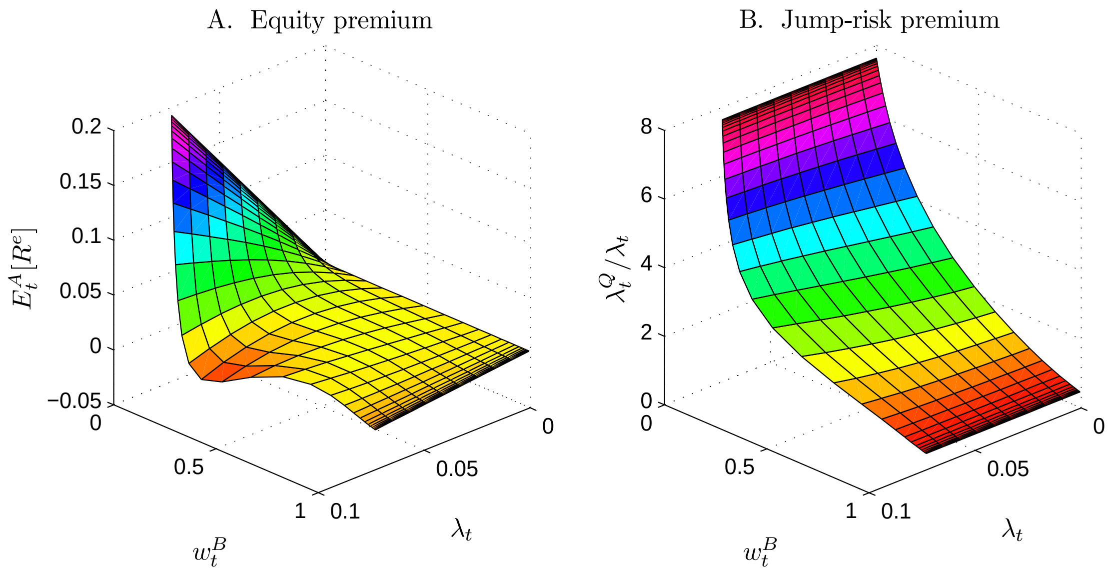

Figure 4: **Time-varying Disaster Risk.** Panel A plots the equity premium under agent A's beliefs as a function of agent B's wealth share (w_t^B) and the disaster intensity under A's beliefs (λ_t). Panel B plots the jump-risk premium $\lambda_t^{\mathbb{Q}}/\lambda_t$ for agent A.

premium becomes essentially flat as λ_t varies. This result has important implications for the time series properties of the equity premium. It suggests that when λ_t fluctuates over time, the equity premium can either be volatile or smooth, depending on the wealth distribution.

We can understand the above results through the equity premium formula,

$$E_t^A[R^e] = \gamma\sigma_c^2 - E_t[\Delta R]\left(\frac{\lambda_t^{\mathbb{Q}}}{\lambda_t} - 1\right)\lambda_t\,, \tag{22}$$

where now the return conditional on a disaster occurring, $E_t[\Delta R]$, does not depend on the probability measure since there is a single disaster type. Variations in the wealth distribution drive $\lambda_t^{\mathbb{Q}}/\lambda_t$ and $E_t[\Delta R]$. Due to increased risk sharing, the jump-risk premium declines with greater fraction of wealth controlled by the optimistic agent. As a result, the premium becomes less sensitive to variations in λ_t. Moreover, we see in Panel B of Figure 4 that the effect of wealth on the jump risk premium depends on the disaster intensity – when the disaster intensity is high, the risk sharing motives are very strong, resulting in larger effect on the jump risk premium when the optimistic agent controls even a small amount of wealth. Finally, the returns in disasters also vary somewhat with the wealth distribution as the price-consumption ratio changes after a disaster.

To further investigate the time series properties of the model, we simulate the disaster intensity λ_t and the jump component of aggregate endowment c_t^d under agent A's beliefs, which jointly determine the evolution of the stochastic Pareto weight $\tilde{\zeta}_t$. Then, along the simulated paths, we compute the equilibrium wealth fraction of agent A, w_t^A, and the conditional equity premium under A's beliefs, $E_t^A[R^e]$. In each simulation we start with $\lambda_0 = 1.7\%$ and set the initial wealth share of agent A to $w_0^A = 90\%$. The results from two of the simulations are reported in Figure 5.

Panel A plots the paths of λ_t from the simulations. The disaster intensities from both simulations are fairly persistent, and show similar amount of variation over time. What are not shown in this graph are the occurrences of disasters. In Simulation I, there are no disasters. In Simulation II, disasters occur three times within the first 50 years, around year 13, 18, and 46.

What determines the evolution of the wealth distribution? When there are no disasters, holding λ_t fixed, agent A is losing wealth share to B as she pays B the premium for disaster insurance. This effect is captured by the negative drift in the Radon-Nikodym derivative η_t (see equation (3)), and is stronger when λ_t^A is larger. In addition, as λ_t falls (rises), the value of the disaster insurance that agent A owns falls (rises), causing her wealth to fall (rise) relative to agent B, who is short the disaster insurance. As Panel B shows, the second effect appears to be the main force driving the wealth distribution in Simulation I.

When a disaster strikes, the wealth distribution can change dramatically. In Simulation II, the wealth share of agent A jumps up each time a disaster strikes. This is because the disaster insurance that A (pessimist) purchases from B (optimist) pays off

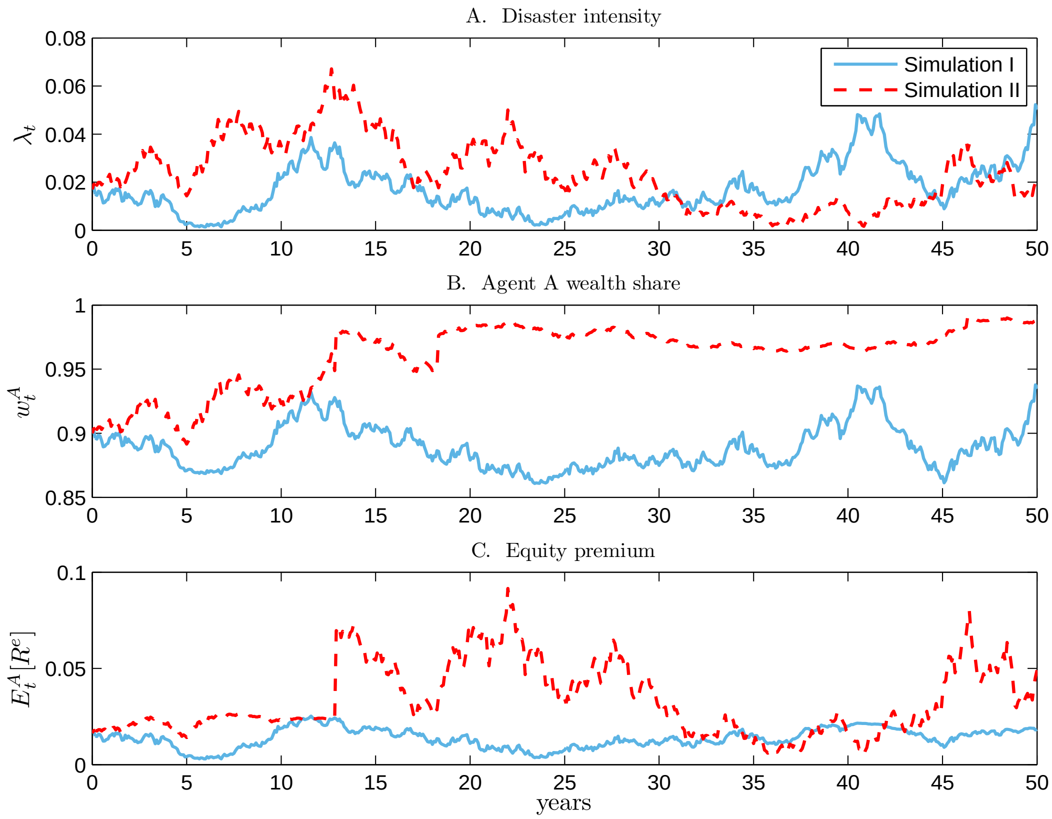

Figure 5: **Simulation with Time-varying Disaster Risk.** The results are from two simulations of the model with time-varying disaster risk under agent A's beliefs. Panel A plots the simulated paths of disaster intensity. Panel B and C plot the corresponding wealth share of agent A and the conditional equity premium she demands.

at such times, causing the wealth of A to increase relative to B. The size of the jump in w_t^A is bigger in the first two disasters, which is due to two reasons. First, during the first two disasters, the wealth distribution is not too concentrated in the hands of agent A, so that agent B can still provide a fair amount of risk sharing. Second, the first disaster occurs at times when λ_t is relatively high, i.e., they are less of a "surprise". Thus, agent A will have bought more insurance against the disaster beforehand, causing her wealth share to rise more after the disaster.

Panel C shows the joint effect of the disaster intensity and wealth distribution on the equity premium. In Simulation I (no disasters), despite the fact that the optimistic agent never owns more than 15% of total wealth and that disaster intensity λ_t shows considerable variation over the period, the equity premium is below 2% nearly 90% of the time. This result confirms our finding in Figure 4 that risk sharing between the agents keeps the premium low and smooth when the wealth share of agent B is not too small. In contrast, the equity premium in Simulation II shows large variation, ranging from 0.5% to 9.2%. Besides becoming significantly more sensitive to fluctuations in λ_t, the premium also changes with the wealth distribution. In particular, the premium jumps up after each disaster. Since the wealth share of agent B drops in a disaster, her risk sharing capacity is reduced, which drives up the equity premium. As show in Figure 4, this effect is stronger when λ_t is high, which is why the jump in premium is most visible after the first disaster (year 13).

4 General Forms of Disagreements

The affine heterogeneous beliefs framework in Section 2 can capture other forms of heterogeneous beliefs besides disagreement about disaster intensity. In this section, we first show that disagreement about the size of disasters has similar impact on the risk premium as disagreement about the frequency of disasters. We then provide an example with strong effects of risk sharing even when both agents are pessimistic about disasters. Finally, we calibrate two sets of beliefs using international and US historical data.[14]

4.1 Disagreement about the Size of Disasters

For simplicity, let's assume that the drop in aggregate consumption in a disaster follows a binomial distribution, with the possible drops being 10% and 40%. Both agents agree on the intensity of a disaster ($\lambda = 1.7\%$). Agent A (pessimist) assigns a 99% probability

[14] The general form of the analysis also generalizes to the case of heterogeneous risk aversion. See Appendix F for details.

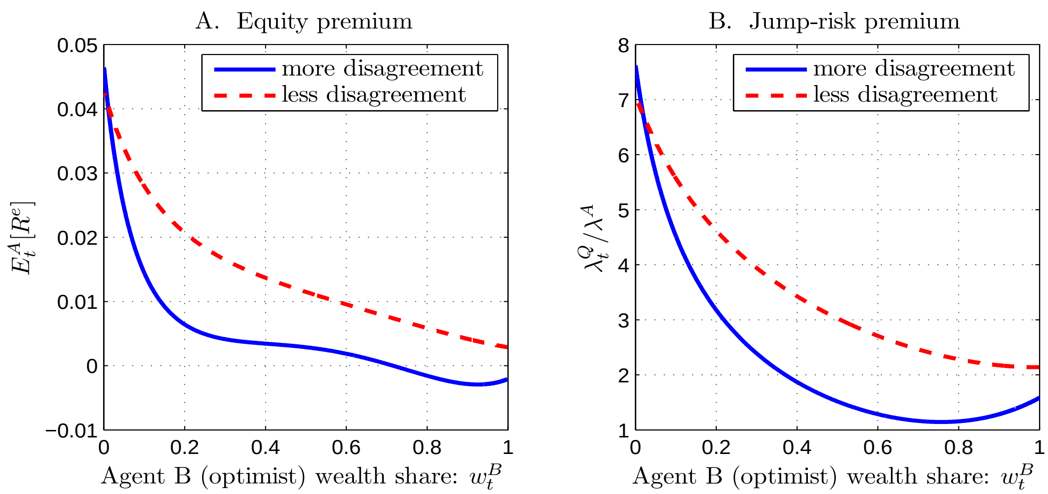

Figure 6: **Disagreement about the size of disasters.** The left panel plots the equity premium under the pessimist's beliefs. The right panel plots the jump-risk premium for the pessimist. In the case with "more disagreement", the pessimist (optimist) assigns 99% probability to the big (small) disaster, conditional on a disaster occurring. With "less disagreement", the probability assigned to big (small) disaster drops to 90%.

to a 40% drop in aggregate consumption, thus having essentially the same beliefs as in the previous example. On the contrary, agent B (optimist) only assigns 1% probability to a 40% drop, but 99% probability to a 10% drop. The rest of the parameter values are the same as in the first example.

Figure 6 (solid lines) plots the conditional equity premium and jump-risk premium under the pessimist's beliefs. When the pessimist has all the wealth, the equity premium is 4.6% (almost the same as in the first example). Again, the equity premium falls rapidly as we starts to shift wealth to the optimist. The premium falls by almost half to 2.4% when the optimist owns just 5% of total wealth, and becomes 1.4% when the optimist's share of total wealth grows to 10%. Similarly, the jump-risk premium falls from 7.6 to 4.5 with the optimist's wealth share reaching 10%, which by itself will lower the premium to 2.4%.

These results show that, in terms of asset pricing, introducing an agent who disagrees about the severity of disasters is similar to having one who disagrees about the frequency of disasters. Even though the two agents agree on the intensity of disasters in general, they actually strongly disagree about the intensity of disasters of a specific magnitude. For example, under A's beliefs, the intensity of a big disaster is $1.7\% \times 99\% = 1.68\%$, which is 99 times the intensity of such a disaster under B's beliefs. The opposite is true for small disasters. Thus, B will aggressively insure A against big disasters, while A insures B against small disasters. For agent A, the effect of the reduction in consumption loss in a big disaster dominates that of the increased loss in a small disaster, which drives down the equity premium exponentially. Such trading can also become speculative when B has most of the wealth: agent A will take on so much loss in a small disaster that the jump-risk premium rises up again.

Naturally, we expect that the agents will be less aggressive in trading disaster insurances when there is less disagreement on the size of disasters, and that the effect of risk sharing on the risk premium will become smaller. The case of "less disagreement" in Figure 6 confirms this intuition. In this case, we assume that the two agents assign 90% probability (as opposed to 99%) to one of the two disaster sizes. While the equity premium still falls rapidly near the left boundary, the pace is slower than in the previous case. Similarly, we see a slower decline in the jump-risk premium.

4.2 When Two Pessimists Meet

The examples we have considered so far have one common feature: the new agent we are bringing into the economy has more optimistic beliefs about disaster risk, in the sense that the distribution of consumption growth under her beliefs first-order stochastically dominates that of the other's, and that the equity premium is significantly lower when she owns all the wealth. However, the key to generating aggressive risk sharing is not that the new agent demands a lower equity premium, but that she is willing to insure the majority wealth holders against the types of disasters that they fear most.

In order to highlight this insight, we consider the following example, which combines disagreements about disaster intensity as well as disaster size. Both agents believe

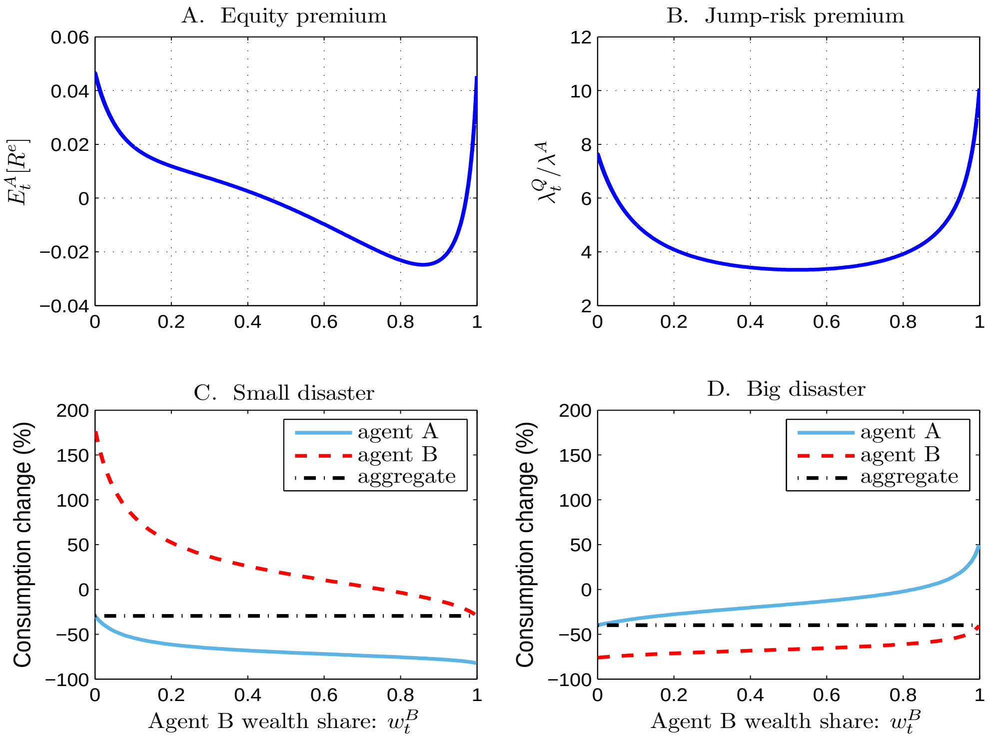

Figure 7: **When Two Pessimists Meet.** Panel A and B plot the equity premium and jump-risk premium under agent A's beliefs. Panel C and D plot the individual consumption changes in small and big disasters.

that disaster risk accounts for the majority of the equity premium. The key difference in their beliefs is that one agent believes that disasters are rare but big, while the other thinks disasters are more frequent but less severe. Specifically, we assume that disasters can cause aggregate consumption drops of a 30% or 40%. Agent A believes that $\lambda^A = 1.7\%$, and assigns 99% probability to the bigger disaster. B believes that $\lambda^B = 4.2\%$, and assigns 99% probability to the smaller disaster.

By themselves, the two agents both demand high equity premium. We have chosen λ^B so that, under the beliefs of agent A, the equity premium is 4.6% whether A or B has all the wealth. However, they have significant disagreement on the exact magnitude of the disaster. Such disagreement generates a lot of demand for risk sharing. As we see in Panel A of Figure 7, the conditional equity premium falls rapidly as the wealth share of agent B moves away from the two boundaries. In fact, the premium will be below 2% when B owns between 9% and 99% of total wealth. In Panel B, the jump risk premium also falls by half from 7.6 and 10 on the two boundaries when B's wealth share moves from 0% to 25% and from 100% to 91%, respectively.

To get more information on the risk sharing mechanism, in Panel C and D we examine the equilibrium consumption changes for the individual agents during a small or big disaster. Since agent A assigns a low probability to the small disaster, she insures agent B against this type of disasters. As a result, her consumption loss in such a disaster exceeds that of the aggregate endowment (-30%), and it increases with the wealth share of agent B. When B has almost all the wealth in the economy, agent A sells so much small disaster insurance to B that her own consumption can fall by as much as 82% when such a disaster occurs. As a result, agent B is able to reduce her risk exposure to small disasters significantly. In fact, her consumption actually jumps up in a small disaster when she owns less than 75% of total wealth, sometimes by over 100% (when her wealth share is small).

The opposite is true in Panel D. As agent B insures A against big disasters, she experiences bigger consumption losses in such a disaster than the aggregate endowment (-40%). The equilibrium consumption changes of the two agents are less extreme compared to the case of small disasters, which is due to two reasons. First, the relative disagreement on big disasters is smaller than on small disasters. Second, the insurance against larger disasters is more expensive, so that agent A's ability to purchase disaster insurance is more constrained by her wealth.

4.3 Calibrating Disagreement: Is the US Special?

Having considered a series of special examples of heterogeneous beliefs, we now extend the analysis to a less stylized model of beliefs on disasters. We calibrate the beliefs of the two types of agents is as follows. Agent A believes that the US is no different from the rest of the world in its disaster risk exposure. Hence her beliefs are calibrated using cross-country consumption data. Agent B, on the other hand, believes that the US is special. She forms her beliefs on disaster risk using only the US consumption data.

An important contribution of Barro (2006) is to provide detailed accounts of the major consumption declines cross 35 countries in the twentieth century. Rather than directly using the empirical distribution from Barro (2006), we estimate a truncated Gamma distribution for the log jump size from Barro's data using maximum likelihood (MLE).[15] Our estimation is based on the assumption that all the disasters in the sample were independent, and that the consumption declines occurred instantly.[16] We also bound the jump size between -5% and -75%. In comparison, the smallest and largest declines in per capital GDP in Barro's sample are 15% and 64%, respectively. The disaster intensity under A's beliefs is still $\lambda^A = 1.7\%$. The remaining parameters are: the mean growth rate and volatility of consumption without a disaster, $\bar{g} = 2.5\%$ and $\sigma_c = 2\%$, which are consistent with the US consumption data post WWII.

As for agent B, we assume that she agrees with the values of $\bar{g}$ and σ_c, but we estimate the truncated Gamma distribution of disaster size using MLE from annual per-capita consumption data in the US 1890-2008.[17] Over the sample of 119 years, there are three years where consumption falls by over 5%. Thus, we set $\lambda^B = 3/119 = 2.5\%$. Alternatively, we can also jointly estimate λ^B and the jump size distribution.

Panel A of Figure 8 plots the probability density functions of the log jump size distributions for the two agents, which are very different from each other. The solid line is the distribution fitted to the international data on disasters. The average log drop is 0.36, which is equivalent to 30% drop in the level of consumption. In the US data, the average drop in log consumption is only 0.075, or 7.3% in level. In addition, agent

[15]The truncated Gamma distribution has PDF $f(d;\alpha,\beta|d_{min},d_{max}) = f(d;\alpha,\beta)/\left(F(d_{max};\alpha,\beta) - F(d_{min};\alpha,\beta)\right)$, where $f(x;\alpha,\beta)$ and $F(x;\alpha,\beta)$ are the PDF and CDF of the standard Gamma distribution with shape parameter α and scale parameter β.

[16]These assumptions are debatable. For example, many of the major declines cross European countries are in WWI and WWII. Moreover, many of the declines spanned several years. See Barro and Ursúa (2008), Donaldson and Mehra (2008), and Constantinides (2008) for more discussions on the measurement of historical disasters.

[17]The data is taken from Robert Shiller's web site http://www.econ.yale.edu/~shiller/data.htm

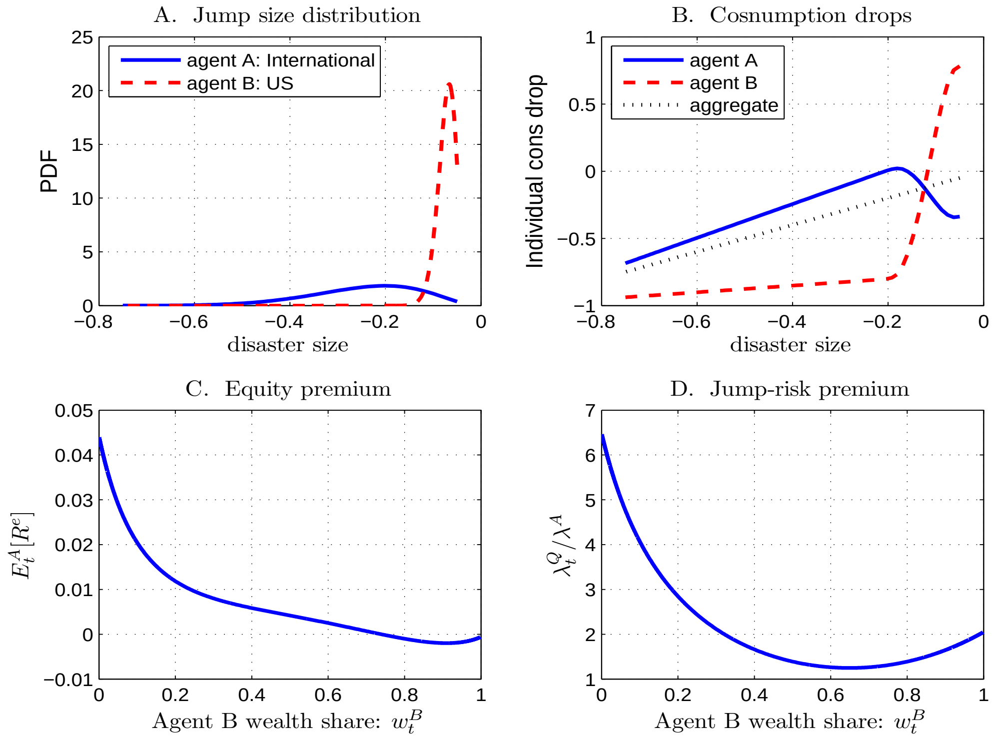

Figure 8: **Calibrated Disagreements: International vs US Experiences.** Panel A plots the truncated Gamma distribution of disaster size for the two agents. Panel B plots the equilibrium consumption drops for the two agents given the size of the disaster. Panel C and D plot the equity premium and jump-risk premium under A's beliefs.

A's distribution has a much fatter left tail than B. Thus, while A assigns significantly higher probabilities than B to large disasters (where consumption drops by 15% or more), agent B assigns more probabilities to small disasters, especially those ranging from 5 to 12%. In fact, agent B's beliefs are close to the calibration adopted by Longstaff and Piazzesi (2004), who assume that the jump in aggregate consumption during a disaster is 10%.

The differences in beliefs lead the two agents to insure each other against the types of disasters they fear more, and the trading can be implemented using a continuum of disaster insurance contracts with coverage specific to the various disaster sizes. Panel B plots drops in the equilibrium consumption (level) for the two agents when disasters of different sizes occur, assuming that agent B owns 10% of total wealth. The graph shows that through disaster insurances, agent A is able to reduce her consumption loss in large disasters (comparing the solid line to the dotted line). For example, her own consumption will only fall by 24% in a disaster where aggregate consumption falls by 40%, a sizable reduction especially considering the small amount of wealth that agent B has. At the same time, she also provides insurances to B on smaller disasters, which increases her consumption losses when such disasters strike. Agent B's consumption changes are close to a mirror image of agent A's. However, the changes are magnified both for large and small disasters due to her small wealth share.

Panel C shows the by-now familiar exponential drop in the equity premium as the wealth share of agent B increases. The equity premium is 4.4% when all the wealth is owned by the agents who form their beliefs about disasters based on international data, but drops to 2.0% when just 10% of total wealth is allocated to the agents who form their beliefs using only the US data. The main reason for the lower equity premium is again due to the decrease of the jump-risk premium (Panel D), which falls from 6.5 to 4.0 when agent B's wealth share rises to 10%. This effect alone drives the equity premium down to 2.4%. Notice that the jump-risk premium is no longer monotonic in the wealth share of agent B. This is because when agent A has little wealth, she would be betting against small disasters so aggressively that the big losses for her during small disasters can cause the jump-risk premium to rise again.

5 Time-varying Disagreement

The results in the previous section not only demonstrate the large impact that risk sharing can have on the equity premium, but also highlight the conditions under which disaster risk matters the most. For example, the equity premium becomes higher and significantly more sensitive to fluctuations in disaster risk λ_t when the pessimistic agent has most of the wealth. Another way to reduce risk sharing is by having the beliefs of the agents converge, which has been ruled out in our model. In reality, investors' beliefs could converge or diverge. In particular, if there is information signaling that the risk of disasters is rising in the economy, it is possible that the optimists will update their beliefs more than the pessimists, so that the difference in beliefs becomes smaller.

In this section, we extend the model from Section 3.1 to capture the effect of time variation in disagreement. We assume the economy can be in one of two states, $s_t = L, H$. In state L, the two agents' perceived disaster intensity are λ_L^A and λ_L^B, while in state H, they become λ_H^A and λ_H^B. The transitions between the two states are governed by a continuous-time Markov chain, with the generator matrix

$$\mathbf{\Lambda} = \begin{bmatrix} -\delta_L & \delta_L \\ \delta_H & -\delta_H \end{bmatrix}.$$

For example, the probability of the economy moving from state L to state H over a short period Δt will be approximately $\delta_L \Delta t$. We assume that the agents agree on the transition probabilities of the Markov chain. Moreover, they agree on the size of disasters (which is constant) as well as the Brownian risk, and have the same preference parameters as in Section 3.1.

The Radon-Nikodym derivative η_t now reflects the change of state s_t,

$$\eta_t = e^{\sum_{i\in\{L,H\}}\left(\Delta a_i N_t^i - \lambda_i^A T_t^i (e^{a_i}-1)\right)}, \tag{23}$$

where

$$\Delta a_i = \log\left(\frac{\lambda_i^B}{\lambda_i^A}\right), \tag{24}$$

$$T_t^i = \int_0^t 1_{\{s_\tau = i\}} d\tau, \tag{25}$$

and N_t^i counts the number of disasters that have occurred up to time t while the state is $s_t = i$.

We solve the planner's problem in a similar way as before. Using the results on the occupation time of continuous-time Markov chains (see e.g., Darroch and Morris (1968)), we derive the price of aggregate consumption claim and the equity risk premium in closed form. The details of the derivation are in Appendix G.

We first analyze the case where beliefs converge (diverge) at times when disaster risk rises (drops). In state L we assume the risk of disasters is low, and the amount of disagreement between the two agents is large. The actual beliefs are $\lambda_L^A = 1.7\%$ and $\lambda_L^B = 0.1\%$, the same as in Section 3.1. In state H, the risk of disasters is higher, while the relative differences in beliefs between agent A and B are smaller. Specifically, we assume that $\lambda_H^A = 2.5\%$ and $\lambda_H^B = 1.25\%$, so that agent A still views disasters twice as likely as agent B does. For the Markov chain, we set $\delta_L = 0.1$ and $\delta_H = 0.5$, so that the high-disaster-risk state is more transitory.

The results are quite intuitive. When there is a 10% probability of moving into a high-disaster-risk state within a year, there is almost no effect on the equity premium in state L. When the economy is in state H, the equity premium rises, especially at times when agent B has a nontrivial share of total wealth. For example, when the economy moves from state L to H, the equity premium agent A demands rises from 4.7% to 7% when B has no wealth. If agent B has 20% of total wealth, the equity premium increases from 1.7% to 5.2%. The rise in premium is in part due to higher disaster risk, as λ^A rises from 1.7% to 2.5%. Another reason is that there is less disagreement between the two agents in state H, as $\lambda_H^A/\lambda_H^B < \lambda_L^A/\lambda_L^B$. Hence, there is less risk sharing between the two agents, and the pessimistic agent will have to bear bigger losses in consumption in a disaster.

Next, we analyze the case where beliefs diverge when disaster risk rises. In this exercise, we assume that there is no disagreement in state L, $\lambda_L^A = \lambda_L^B = 1.7\%$. The beliefs in state H satisfy $(1 - w^B)\lambda_H^A + w^B\lambda_H^B = 1.7\%$, where w^B is the wealth share of agent B. Thus, as we increase the disagreement about disaster intensity between the two agents in state H, the wealth-weighted average belief remains the same. We

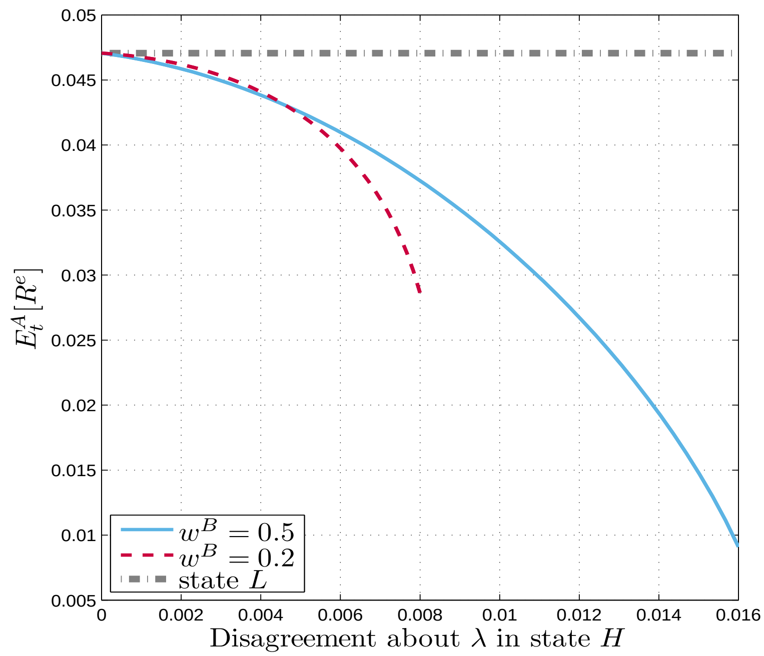

Figure 9: **Time-varying Disagreement.** Panel A plots the equity premium in the case where beliefs converge in the state with higher disaster risk. Panel B plots the premium as a function of the amount of disagreement for given wealth distribution.

measure the the amount of disagreement with the standard deviation in beliefs,

$$\text{Disagreement Measure} = \sqrt{(1 - w^B)(\lambda_H^A - 1.7\%)^2 + w^B(\lambda_H^B - 1.7\%)^2}.$$

Again, we set the transition probabilities of the Markov chain to be $\delta_L = 0.1$ and $\delta_H = 0.5$.

Figure 9 shows, holding the average belief constant, the premium can fall substantially as the amount of disagreement increases. As a benchmark, the dash-dotted line gives the equity premium (under agent A's beliefs) in state L. Since the agents have the same beliefs in that state, the premium remains at 4.7% as the amount of disagreement increases in state H. The solid line plots the equity premium in state H when the two agents have equal share of total wealth. The premium falls from 4.7% to 0.9% when λ_H^B drops from 1.7% to 0.1% (where the disagreement measure is 0.016). When agent

B has just 20% of total wealth, the premium falls by a smaller amount to 2.9% (when the disagreement measure reaches 0.008). An interesting implication of this graph is that the premium can actually be *decreasing* while the average belief of disaster risk *increases*, provided that there is enough increase in the amount of disagreement at the same time.

In summary, besides the variation in disaster risk and wealth distribution across agents with heterogeneous beliefs, time variation in the amount of disagreement across agents can be another importance source of fluctuations in disaster risk premium.

6 Concluding Remarks

We demonstrate the equilibrium effects of reasonable disagreement about disasters on risk premia and trading activities. When agents disagree about disaster risk, they will insure each other against the types of disasters they fear most. Because of the highly nonlinear effect of disaster size on risk premia, the risk sharing provided by a small amount of agents with heterogeneous beliefs can significantly attenuate the effect of disasters on the equity premium. The model also has several important implications for the dynamics of asset prices.

We should emphasize that our results do not necessarily diminish the importance of disaster risk for the equity premium. The effectiveness of risk sharing hinges on complete markets. The amount of disaster insurance being traded in our model, while still within the limit imposed by the budget constraint, can be difficult to implement in practice due to moral hazard. Even exchange trading and daily mark-to-market will not eliminate the counterparty risks associated with these contracts without large collateral constraints, because disasters will lead to sudden large changes in prices. From this perspective, our results highlight the importance of incorporating market incompleteness in disaster risk models. It would be very useful to study what happens to asset prices when we limit the risk sharing among investors with heterogeneous beliefs about disasters, perhaps by imposing transaction costs, borrowing constraints, and short-sales constraints[18] as in Heaton and Lucas (1996).

Another possible way to reduce the effects of heterogeneous beliefs is through ambiguity aversion. As Hansen (2007) and Hansen and Sargent (2009) show, if investors are ambiguity averse, they deal with model/parameter uncertainty by slanting their beliefs pessimistically. In the case with disaster risk, confronting investors with the same model uncertainty facing econometricians could lead them to behave as if they believe the disaster probabilities are high, even though their actual priors might suggest otherwise. This mechanism could reduce the heterogeneity of the distorted beliefs among agents, thus limiting the effects of risk sharing. We leave these implications to future research.

[18]Since the primary risk in the aggregate endowment claim is disaster risk, shorting the stock might serve as a close substitute to buying disaster insurance.

Appendix

A Securities' prices and portfolio positions

In this appendix we compute the prices of the claim on aggregate endowment (stock), the claim on individual agents' consumption streams (agents' personal wealth), disaster insurance, and the equilibrium portfolio positions. We begin with the general setting of time-varying disaster intensity. To concentrate on the effects of heterogeneous beliefs, we assume that the two agents have the same relative risk aversion γ.

A.1 Aggregate and individual consumption claim prices: general setting

The price of the aggregate endowment claim is

$$P_t = \int_0^\infty E_t^{\mathbb{P}_A}\left[\frac{M_{t+T}^A}{M_t^A} C_{t+T}\right] dT\,, \tag{A.1}$$

where M_t^A is the stochastic discount factor

$$M_t^A = e^{-\rho t} C_t^{-\gamma}\left(1 + (\zeta_0 e^{\log \eta_t})^{\frac{1}{\gamma}}\right)^{\gamma}\,. \tag{A.2}$$

This price can be viewed as a portfolio of zero coupon aggregate consumption claims

$$\begin{aligned} M_t^A P_t^{t+T} &= E_t^{\mathbb{P}_A}[M_{t+T}^A C_{t+T}] \\ &= e^{-\rho(t+T)} e^{T[\bar{g}(1-\gamma)+\frac{1}{2}\sigma_c^2(1-\gamma)^2]} e^{(1-\gamma)c_t} \times E_t^{\mathbb{P}_A}\left[e^{(1-\gamma)c_{t+T}^d}\left(1 + (\zeta_0 e^{\log \eta_{t+T}})^{\frac{1}{\gamma}}\right)^{\gamma}\right]. \end{aligned}$$

Under our assumption of integer γ, the final term will be a sum of expectations of the form

$$E_t^{\mathbb{P}_A}[e^{(1-\gamma)c_{t+T}^d + \beta_i \log \eta_{t+T})}] = e^{A_i(T)+(1-\gamma)c_t^d+\beta_i \log \eta_t + B_i(T)\lambda_t}, \tag{A.3}$$

where (A_i, B_i) satisfy a simplified version of the familiar Riccati differential equations

$$\dot{B}_i = -\frac{\bar{\lambda}^B}{\bar{\lambda}^A}\beta_i - \kappa B_i + \frac{\sigma_\lambda^2}{2}B_i^2 + (\phi(\langle 1-\gamma, \beta_i\rangle) - 1)\,, \quad B_0(0) = 0\,, \tag{A.4a}$$

$$\dot{A}_i = \kappa\theta B_i\,, \quad A_i(0) = 0\,, \tag{A.4b}$$

where ϕ is the moment generating function of jumps in $\langle c_t^d, a_t\rangle$.

It follows that price/consumption ratio of the zero-coupon equity varies only with the stochastic weight $\tilde{\zeta}_t$ and the disaster intensity:

$$P_t^{t+T} = C_t h^T(\lambda_t, \tilde{\zeta}_t)\,. \tag{A.5}$$

Next, agent A's wealth $P_t^A = \int_0^\infty E_t^{\mathbb{P}_A}\left[\frac{M_{t+T}^A}{M_t^A} C_{t+T}^A\right] dT$ at time t is a portfolio of her zero coupon consumption claims

$$\begin{aligned} M_t^A P_t^{A,t+T} &= E_t^{\mathbb{P}_A}[M_{t+T}^A C_{t+T}^A] \\ &= e^{-\rho(t+T)} e^{T[\bar{g}(1-\gamma)+\frac{1}{2}\sigma_c^2(1-\gamma)^2]} e^{(1-\gamma)c_t} \times E_t^{\mathbb{P}_A}\left[e^{(1-\gamma)c_{t+T}^d}\left(1+(\zeta_0 e^{\log \eta_{t+T}})^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]. \end{aligned}$$

We can compute agent A's wealth process by making a similar binomial expansion as in the case of P_t, and then computing the expectation concerning the same affine jump diffusion process. Finally, the wealth process of agent B is simply $P_t^B = P_t - P_t^A$.

A.2 Special case: constant disaster risk

Closed form expressions can now be obtained in the special case of constant disaster intensity and constant disaster size. Let's denote $\tilde{\zeta}_t \equiv \zeta_0 e^{\log \eta_t}$. Again by expanding the binomial for the cases with integer γ,

$$\begin{aligned} E_t^{\mathbb{P}_A}\left[M_{t+T}^A C_{t+T}\right] &= e^{-\rho(t+T)} E_t^{\mathbb{P}_A}\left[\left(1+(\tilde{\zeta}_{t+T})^{1/\gamma}\right)^{\gamma} C_{t+T}^{1-\gamma}\right] \\ &= e^{-\rho(t+T)} C_t^{1-\gamma} \sum_{k=0}^{\gamma} \binom{\gamma}{k} E_t^{\mathbb{P}_A}\left[\frac{(\tilde{\zeta}_{t+T})^{k/\gamma} C_{t+T}^{1-\gamma}}{C_t^{1-\gamma}}\right]. \end{aligned}$$

Plugging in the explicit expressions for aggregate consumption C_t, the stochastic discount factor M_t^A, and performing the simple affine jump diffusion expectation we obtain

$$P_t^{t+T} = C_t \sum_{k=0}^{\gamma} \alpha_{k,t} e^{-\beta_k T}\,, \tag{A.6}$$

with

$$\alpha_{k,t} \equiv \binom{\gamma}{k} \frac{(\tilde{\zeta}_t)^{k/\gamma}}{(1+(\tilde{\zeta}_t)^{1/\gamma})^{\gamma}}\,, \tag{A.7a}$$

$$\beta_k \equiv \rho + (\gamma-1)\bar{g} - \frac{1}{2}\sigma_c^2(\gamma-1)^2 - \bar{\lambda}(e^{(\gamma-1)\bar{d}+\frac{k\Delta a}{\gamma}} - 1) + \frac{\bar{\lambda}k}{\gamma}(e^{\Delta a} - 1)\,, \tag{A.7b}$$

where Δa is given in (5).

Finally, integrating over time T yields the explicit price of aggregate endowment claim

$$P_t = \int_0^\infty P_t^{t+T} dT = C_t \sum_{k=0}^{\gamma} \frac{\alpha_{k,t}}{\beta_k} \, . \tag{A.8}$$

The restriction $\beta_k^A > 0$ is needed to ensure finite value for P_t. We will come back to this type of restriction below.

By identical approach, we obtain the price of agent A's consumption claim (i.e. her wealth process)

$$P_t^A = \int_0^\infty P_t^{A,t+T} dT = C_t \sum_{k=0}^{\gamma-1} \frac{\alpha_{k,t}^A}{\beta_k} \, , \tag{A.9}$$

where β_k remains the same as above and

$$\alpha_{k,t}^A \equiv \begin{pmatrix} \gamma - 1 \\ k \end{pmatrix} \frac{(\tilde{\zeta}_t)^{k/\gamma}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} \, . \tag{A.10}$$

Price of disaster insurance

Let $P_{t,t+T}^{DI}$ denotes the price of disaster insurance which pays \$1 at maturity time $t+T$ if there was at least one disaster taking place in the time interval $(t, t+T)$. In the main text we consider disaster insurance P_t^{DI} of maturity $T = 1$ in particular.

$$\begin{aligned} P_{t,t+T}^{DI} &= E_t^{\mathbb{P}_A} \left[\frac{M_{t+T}^A}{M_t^A} \mathbf{1}_{(N_{t+T} > N_t)} \right] \\ &= \frac{e^{-\rho T}}{(C_t^A)^{-\gamma}} E_t^{\mathbb{P}_A} \left[(C_T^A)^{-\gamma} \mathbf{1}_{(N_{t+T} > N_t)} \right] \\ &= \frac{e^{(-\rho - \gamma \bar{g} + \frac{1}{2}\gamma^2 \sigma_c^2)T}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} E_t^{\mathbb{P}_A} [e^{\gamma \bar{d} \Delta N_T} (1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(\Delta a \Delta N_T - \bar{\lambda} T (e^{\Delta a} - 1))/\gamma})^\gamma \mathbf{1}_{(\Delta N_T > 0)}] \\ &= \frac{e^{(-\rho - \gamma \bar{g} + \frac{1}{2}\gamma^2 \sigma_c^2)T}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} \{ E_t^{\mathbb{P}_A} [e^{\gamma \bar{d} \Delta N_T} (1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(\Delta a \Delta N_T - \bar{\lambda} T (e^{\Delta a} - 1))/\gamma})^\gamma] \\ &- (1 + (\tilde{\zeta}_t)^{1/\gamma} e^{-\bar{\lambda} T (e^{\Delta a} - 1)/\gamma})^\gamma \mathbb{P}_A(\Delta N_T = 0) \} \, , \end{aligned}$$

where $\Delta N_T \equiv N_{t+T} - N_t$ is number of disasters taking place in $[t, t+T]$, and $\mathbb{P}_A(\Delta N_T = 0) = e^{-\bar{\lambda} T}$ is the probability that no such disaster did happen. Again by expanding the binomial $(1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(\Delta a \Delta N_T - \bar{\lambda} T (e^{\Delta a} - 1))/\gamma})^\gamma$, and then computing the expectation of each resulting term, we obtain

$$P_{t,t+T}^{DI} = \frac{a_T}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} \{ [\sum_{k=0}^{\gamma} b_{k,T} (\tilde{\zeta}_t)^{k/\gamma}] - e^{-\bar{\lambda} T} (1 + (\tilde{\zeta}_t)^{1/\gamma} e^{-\bar{\lambda} T (e^{\Delta a} - 1)/\gamma})^\gamma \} \, , \tag{A.11}$$

where

$$a_T = e^{(-\rho-\gamma\bar{g}+\frac{1}{2}\gamma^2\sigma_c^2)T}\,, \tag{A.12a}$$

$$b_{k,T} = \begin{pmatrix} \gamma \\ k \end{pmatrix} e^{-\bar{\lambda}kT(e^{\Delta a}-1)/\gamma} e^{\bar{\lambda}T[e^{(\gamma d+\frac{\Delta a k}{\gamma})}-1]}\,. \tag{A.12b}$$

B Equilibrium portfolio positions

In the current case of constant jump size with two dimensions of uncertainties (Brownian motion and disaster jump), the market is complete when agents are allowed to trade contingent claims on aggregate consumption (stock) P_t, money market account RFB_t and disaster insurance P_t^{DI}. We can use generalized Ito lemma on jump-diffusion (see, for example, Protter (2003)) to determine the price processes for each asset. Portfolio positions are then determined by equating the exposures to the Brownian and jump risks of each agents consumption claim to a portfolio of the aggregate claim and disaster insurance, which are then financed with the risk free bond.

C Boundedness of prices

This appendix discusses the boundedness of securities prices in general heterogeneous-agent economy. As claimed in the main text, as long as agents have different but equivalent beliefs, necessary and sufficient condition for finite price of a security in heterogeneous-agent economy is that this price be finite under each agent's beliefs in a single-agent economy. This is easy to see since

$$\max(f_{A,0}^{\gamma}, f_{B,0}^{\gamma}\eta_t) \le M_t^A \le (2f_{A,0}^{\gamma}) + (2f_{B,0}^{\gamma})\eta_t \tag{C.1}$$

Conditions for the finiteness of prices in the single agent economy can be found by studying the fixed points of the equations (A.4a). Setting $dB/dt = 0$, we find the fixed point of this differential equation is

$$B^* = \frac{\kappa - \sqrt{\kappa^2 + 2\sigma_\lambda^2(1-\phi^{\mathbb{P}_i}(1-\gamma^i))}}{\sigma_\lambda^2}\,, \tag{C.2}$$

provided that (13a) holds. Otherwise there is no fixed point and $B \to \infty$ implying infinite prices. Furthermore, it is easily seen that the initial condition $B(0) = 0$ is in the domain of attraction. For equity price to be finite, it is easy to see that the limiting exponent in (A.3) must be negative, or

$$-\rho + (1-\gamma^i)\bar{g} + \frac{1}{2}(\gamma^i-1)^2\sigma_c^2 + \kappa\bar{\lambda}^i B^* < 0\,, \tag{C.3}$$

for both $i = 1, 2$. This is (13b) after we plug in the above expression for B^*.

D Proofs from Section 3.2

In this section, we provide the proofs for (17) and (20). It is useful to rewrite expression for the consumption fractions in terms of the initial consumption sharing rule (f_0^A, f_0^B) and the Radon-Nikodym derivative (η_t). In these terms,

$$\begin{aligned} f_t^A &= \frac{f_0^A}{f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}}}, \\ M_t^A / M_0^A &= \left(f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}} \right)^\gamma C_t^{-\gamma} / C_0^{-\gamma}, \\ \lambda_t^{\mathbb{Q}} &= \lambda_A e^{-\gamma d} \left(f_t^A + f_t^B \left(\frac{\lambda_B}{\lambda_A} \right)^{\frac{1}{\gamma}} \right)^\gamma . \end{aligned}$$

Additionally, for ease of notation, we set $N_0 = 0$ and $C_0 = 1$ which results in the expressions being fractions of the initial endowment.

Taking derivatives, we find

$$\frac{\partial \lambda^{\mathbb{Q}}}{\partial f_0^A} = \lambda_A e^{-\gamma d} \gamma \left(f_0^A + f_0^B \left(\frac{\lambda_B}{\lambda_A} \right)^{\frac{1}{\gamma}} \right)^{\gamma - 1} \left(1 - \left(\frac{\lambda_B}{\lambda_A} \right)^{\frac{1}{\gamma}} \right).$$

Setting $f_0^A = 1$ and taking the limit $\lambda^B \to 0^+$, we obtain (17).

In order to compute the derivative of the wealth fraction of Agent B with respect to f_0^B, we first compute the derivative of the value of his claim, call it P^B, with respect to f_0^B. Since

$$P^B = \int_0^\infty E_0^{\mathbb{P}_A} [(f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}})^{\gamma - 1} f_0^B \eta_t^{\frac{1}{\gamma}} C_t^{1-\gamma}] e^{-\rho t} dt,$$

we have that

$$\begin{aligned} \frac{\partial P^B}{\partial f_0^A} = &\int_0^\infty (\gamma - 1) E_0^{\mathbb{P}_A} [(f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}})^{\gamma - 2} (1 - \eta_t^{1/\gamma}) f_0^B \eta_t^{\frac{1}{\gamma}} C_t^{1-\gamma}] e^{-\rho t} dt \\ &- \int_0^\infty E_0^{\mathbb{P}_A} [(f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}})^{\gamma - 1} \eta_t^{\frac{1}{\gamma}} C_t^{1-\gamma}] e^{-\rho t} dt. \end{aligned}$$

From which it follows

$$\left.\frac{\partial P^B}{\partial f_0^A}\right|_{f_0^A=1} = -\int_0^\infty E_0^{\mathbb{P}_A}[\eta_t^{1/\gamma} C_t^{1-\gamma}] e^{-\rho t} dt$$
$$= -\frac{1}{\rho + (\gamma-1)\bar{g} - \frac{1}{2}\sigma_c^2(1-\gamma)^2 + \frac{1}{\gamma}(\lambda_B - \lambda_A) - \lambda_A(e^{(1-\gamma)\bar{d} + \frac{1}{\gamma}\log\left(\frac{\lambda_B}{\lambda_A}\right)} - 1)}.$$

And so

$$\left.\frac{\partial P^B}{\partial f_0^A}\right|_{f_0^A=1} \to -\frac{1}{\rho + (\gamma-1)\bar{g} - \frac{1}{2}\sigma_c^2(1-\gamma)^2 + \frac{\gamma-1}{\gamma}\lambda_A} \quad \text{as } \lambda_B \to 0^+.$$

Now, it is easy to see that the derivative of the value of the claim to the entire endowment is bounded and since $P_B = 0$ when $f_0^A = 1$, the derivative $\partial w_0^B / \partial f_0^A$ is simply $\frac{\partial P^B}{\partial f_0^A}$ divided by the value of the claim to the entire endowment. This proves (20).

E General valuation of disaster states

In Section 3.2, we demonstrated that within a simple calibration a large fraction of the the value of the endowment claim arises from the disaster states, even though these states are very rare. Here we demonstrate that in fact this property is a feature of a broad class of models. Specifically, suppose that the model is such that the dynamics of aggregate consumption under the actual measure, as well as the risk-neutral measure, follow the dynamics in 1 and that the risk-free rate is constant. This is true in our model with CRRA preferences and remains true with Epstein-Zin preferences (cf. Wachter (2009).) In particular, this reduced form setting removes the link between risk aversion and elasticity of intertemporal substitution.

Within this setting, let $\bar{g}^{\mathbb{Q}}$ denote the growth rate of consumption under the risk neutral measure. The fractional value of consumption in the non-disaster states is then

$$\frac{\int_0^\infty E_0^{\mathbb{Q}}\left[e^{-rt} C_t \times 1_{\{N_t=0\}}\right]}{\int_0^\infty E_0^{\mathbb{Q}}\left[e^{-rt} C_t\right]} = \frac{r - \bar{g}^{\mathbb{Q}} - .5\sigma_c^2 - \lambda^{\mathbb{Q}}(e^{\bar{d}} - 1)}{r - \bar{g}^{\mathbb{Q}} - .5\sigma_c^2 + \lambda^{\mathbb{Q}}}$$

The difference between the numerator and denominator is $\lambda^{\mathbb{Q}} e^{\bar{d}}$. In order for disasters to account for a substantial risk premium, this term should be sizeable (it is 6% in the example of Section 3.1.) Moreover, it is reasonable to expect the price-consumption ratio (the inverse of the denominator) should not be too small. Setting these to 4% and 10 gives a fraction 4/14 due to disaster states. Setting them to 6% and 20 give a fraction of 6/11 to the disaster states. In summary, under these very general reduced form assumptions on the endowment and preferences along with the assumptions that (i) disasters account for a significant risk premium and (ii) the price-consumption ratio

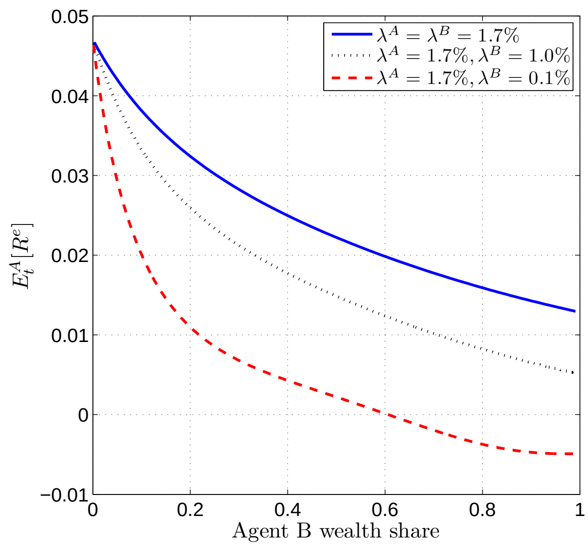

Figure 10: **The effects of heterogeneous risk aversion.** This graph plots the equity premium when the two agents have different risk aversion: $\gamma_A = 4, \gamma_B = 2$. Their beliefs about disasters are specified in the legend. Disaster size is constant.

is not too small, the fraction of wealth due to non-disaster states is significant.[19]

F Heterogeneous Risk Aversion

In this section, we compare our results to models of heterogeneous preferences. Intuitively, besides heterogeneous beliefs, heterogeneity in risk aversion should also be able to induce risk sharing among agents and reduce the equity premium in equilibrium. Recall that the jump-risk premium is $\lambda_t^{\mathbb{Q}}/\lambda_t^i = e^{-\gamma_i \Delta c_t^i}$, which is not only sensitive to changes in individual consumption loss Δc_t^i, but also to the relative risk aversion γ_i. Thus, we expect that heterogeneous risk aversion can have similar effects on the equity premium as heterogeneous beliefs about disasters.

To check this intuition, we consider the following special case of the model. Agent A is the same as in the example of Section 3.1: $\lambda^A = 1.7\%$, $\gamma_A = 4$. Agent B has identical beliefs about disasters but is less risk averse: $\lambda^B = 1.7\%$, $\gamma_B < \gamma_A$. Figure 10 plots the equity premium as a function of agent B's wealth share for $\gamma_B = 2$. The equity premium does decline as agent B's wealth share rises. However, the decline is slow and closer to being linear. In order for the equity premium to fall below 2%,

[19]In the CRRA version of this equation, $r = \rho + \gamma \bar{g} - .5\sigma_c^2\gamma^2 - (\lambda^{\mathbb{Q}} - \lambda^{\mathbb{P}})$. This causes increasing $\lambda^{\mathbb{P}}$ (and thus $\lambda^{\mathbb{Q}}$) to increase the price-consumption ratio. In the general formula if we fix r and increase $\lambda^{\mathbb{Q}}$ independently this *decreases* P/C so clearly the generic form dont have EIS-risk aversion link problems.

the wealth share of the less risk-averse agent needs to rise to 60%. The decline in the equity premium becomes faster as we further reduce the risk aversion of agent B (not reported here), but the non-linearity is still less pronounced than in the cases with heterogeneous beliefs.

Combining heterogeneous beliefs about disasters and different risk aversion can amplify risk sharing and accelerate the decline in the equity premium. As shown in the figure, if agent B believes disasters are less likely than does agent A, and she happens to be less risk averse, the equity premium falls faster. Consider the case where agent B believes disasters only occur once every hundred years ($\lambda^B = 1.0\%$). With 20% of total wealth, she drives the equity premium down by almost a half to 2.5%. If $\lambda^B = 0.1\%$, the decline in the equity premium will be even more dramatic.

G Time-varying Disagreement

The model solution is generally analogous to the case without Markov regime-switching, so we sketch the major differences between the models.

The key expectations to compute are of the form

$$E_0^{\mathbb{P}_A}[e^{aN_t^L+bN_t^H+cT_t^L+dT_t^H}], \tag{G.1}$$

where N_t^i is the number of disasters that occur in state i and T_t^i is the occupation time in state i defined in (25). These expectations can be computed by first conditioning on the path of the Markov state and using the conditional independence of the Poisson process in each state:

$$E_0^{\mathbb{P}_A}[e^{a_L N_t^L+b_H N_t^H+cT_t^L+dT_t^H}] = E_0^{\mathbb{P}_A}\left[E_0^{\mathbb{P}_A}[e^{a_L N_t^L+b_H N_t^H+cT_t^L+dT_t^H}|\{S_\tau\}_{\tau=0}^t]\right] \tag{G.2}$$

$$= E_0^{\mathbb{P}_A}\left[e^{(\lambda_L^A(e^a-1)+c)T^L+(\lambda_H^A(e^b-1)+d)T^H}\right] \tag{G.3}$$

This reduces the problem to computing the joint moment-generating function of the occupation times (T_t^L, T_t^H). Darroch and Morris (1968) show that this expectation reduces to

$$E_0^{\mathbb{P}_A}[e^{\alpha T_t^L+\beta T_t^H}] = \pi_0' \exp(At)\,\vec{1}, \quad \text{where } A = \Lambda + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \tag{G.4}$$

andπ_0 is either $(1,0)'$ or $(0,1)'$, as the initial state is L or H.

The price of consumption claims involve sums of integrals of such expectations. These integral can be computed in closed form by diagonalizing A to deliver closed form expressions for the prices of interest.

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