Idiosyncratic Cash Flows and Systematic Risk[*]

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ABSTRACT

We show that unpriced cash flow shocks contain information about future priced risk. A positive idiosyncratic shock decreases the sensitivity of firm value to priced risk factors and simultaneously increases firm size and idiosyncratic risk. A simple model can therefore explain book-to-market and size anomalies, as well as the negative relation between idiosyncratic volatility and stock returns. Modeling idiosyncratic shocks can also produce a negative relation between growth options and risk and has additional asset pricing implications. More generally, our results imply that any economic variable correlated with the history of idiosyncratic shocks can help to explain expected stock returns.

It is well established that, under standard asset pricing assumptions, only systematic cash flow risk is priced. In this paper, we argue that unpriced idiosyncratic cash flow shocks can also be important for asset prices since they contain valuable conditioning information in a dynamic asset pricing framework. In particular, we show that the conditional beta with respect to any priced source of risk directly depends on the history of firm-specific shocks. We use this insight to provide risk-based explanations for several anomalies in the cross-section of equity returns, including the widely documented value and size effects, the negative relation between idiosyncratic volatility and stock returns, and the underperformance following investment and equity issuance.[1]

To understand why firm-specific shocks are useful as conditioning information, consider a firm with two divisions. Suppose the profit of the first division depends exclusively on idiosyncratic profitability shocks and the profit of the second division is driven only by systematic shocks. This firm can be viewed as a portfolio of a zero-beta asset and a risky asset. When a positive idiosyncratic shock occurs, the size of the zero-beta asset increases, making it a larger fraction of the total portfolio value. As a result, overall firm beta decreases, as do expected stock returns. Therefore, any firm characteristic correlated with the history of idiosyncratic cash flow shocks can help to explain expected stock returns.

In a more general setting, we show that beta is invariant with respect to idiosyncratic shocks only in the special case where profits are the product of idiosyncratic and systematic profitability shocks. Multiplicative value functions of this type are used extensively in the literature because of their tractability properties (see, e.g., Gomes, Kogan, and Zhang (2003), Carlson, Fisher, and Giammarino (2004), Zhang (2005), and Cooper (2006)). Therefore, without additional features such as operating leverage or time-varying price of risk, market betas derived in these studies are independent of firm-specific shocks.[2]

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[1] In the cross-section, firms with small market capitalization and a high ratio of fundamentals to price tend to have high stock returns (Banz (1981), Graham and Dodd (1934)). Fama and French (1992) provide a detailed analysis of both value and size premium. Ang, Hodrick, Xing, and Zhang (2006) document that high idiosyncratic volatility predicts low returns. Among others, Loughran and Ritter (1995), Daniel and Titman (2006), and Pontiff and Woodgate (2008) show that stocks underperform following equity issuances.

[2] Two earlier studies that find dependence of beta on idiosyncratic shocks are Brennan (1973) and Bossaerts and Green (1989), who model cash flows as conditionally linear in a systematic factor and allow the firm-specific intercept to vary over time.

Using this insight, we build a simple model in which firm value is additive in two types of shocks and only systematic risk is priced. We first consider a firm consisting entirely of assets in place, and later add growth options and product market competition. We show that in the simple benchmark model, firm characteristics are related to expected returns in the cross-section. All else being equal, firms with larger idiosyncratic cash flows have larger market capitalizations and lower book-to-market ratios, but at the same time have lower equity betas. As a result, firms with high market capitalization and a low book-to-market ratio have low expected returns.

Similarly, we obtain a negative relation between idiosyncratic volatility and expected stock returns, a puzzling empirical finding documented by Ang, Hodrick, Xing, and Zhang (2006) that seems to be at odds with risk-based explanations. In our one-factor model, a history of favorable idiosyncratic shocks decreases the relative magnitude of the systematic profit component, thereby increasing idiosyncratic stock return volatility and lowering beta. Importantly, idiosyncratic risk is not priced in our framework, but it is correlated with systematic risk and can therefore predict returns in the data.

The model with systematic and idiosyncratic cash flow shocks also adds to our understanding of the relation between growth options and risk. Since investment options are levered claims on assets in place, they are often considered more risky than installed capital. While existing literature shows that this relation can reverse in the presence of operating leverage and adjustment costs (see, among others, Zhang (2005), Carlson, Fisher, and Giammarino (2004), Cooper (2006), and Novy-Marx (2011)), we demonstrate that it also depends on the type of investment options. In particular, while growth options linked to systematic shocks increase a firm's risk, growth options linked to idiosyncratic profitability shocks have the opposite effect. Somewhat surprisingly, however, exercises of both systematic and idiosyncratic growth options always lead to a decline in a firm's systematic risk if the firm finances new investment with equity. Thus, our model also accounts for the observed poor stock return performance following seasoned equity offerings (Loughran and Ritter (1995)).

Further, growth options magnify the value and size premia in the model and give rise

to time-varying price-earnings ratios. There are two reasons for these effects. First, options make firm values and conditional betas more sensitive to profitability shocks, as irreversible investment options grow in value exponentially (Dixit and Pindyck (1994)). Second, firms optimally exercise their investment options, and as a result lower risk, only when their market capitalization is high. We show that the nonlinear exposure of growth options to the underlying profitability shock can generate price-earnings ratios that negatively predict returns.

A generalized version of our model also provides new asset pricing implications of product market competition. As economic conditions improve, greater industry competition leads to an endogenous limit to growth in the systematic component of the profits. When systematic profitability is relatively high, existing firms have low conditional betas because any further increase in profitability is absorbed by newly entering firms. Competition thus adds a dynamic component to our model, resulting in a low risk premium in "good times". This is consistent with empirical evidence provided by Bustamante and Donangelo (2012) that firms in highly competitive industries have low returns. Since all factor betas decline when competition is more intense, the value and size effects also become less pronounced. This result is in agreement with empirical evidence provided by Hou and Robinson (2006), who find that the value effect is stronger in highly concentrated industries.

The intuition developed in this paper applies to any general setting with a single or multiple sources of priced risk. As in previous studies, size and value effects are not anomalous relative to the correctly specified asset pricing model and appear only when not all sources of priced risk are correctly accounted for, as in Berk (1995). Reconciling the predictions of our model with the empirical evidence on value, size, and idiosyncratic volatility anomalies relative to the CAPM thus relies on imperfect measurement of risk, and in particular on differences between conditional and unconditional betas, as in Gomes, Kogan, and Zhang (2003), Carlson, Fisher, and Giammarino (2004), Zhang (2005), and Cooper (2006). Lewellen and Nagel (2006) argue that the conditional CAPM cannot match the magnitude of observed anomalies because the variation in estimated betas is not sufficiently large. However, betas are likely to be mismeasured either because asset pricing tests fail to use all conditioning

information (Hansen and Richard (1987)) or because the proxy for the market portfolio is imperfect (Roll (1977)).

We use the analytical solutions from the model to simulate firms' stock returns and examine the fit between the model-generated and empirically observed data. Our analysis of the simulated data indicates that the model can produce reasonable value and size effects in the cross-sectional Fama and MacBeth (1973) regressions even when we explicitly control for empirically estimated betas. For example, we find a value premium of 56 basis points per month for the decile of largest book-to-market ratios relative to the smallest decile. Sorting based on market capitalization and idiosyncratic volatility yields return differentials of similar magnitudes. Additionally, the decile of high price-earnings ratios underperforms the lowest decile by 17 basis points. Using the simulated data, we show that value and size anomalies are more pronounced in firms with highly valuable growth options and are reduced by product market competition. These results are consistent with empirical evidence in Da, Guo, and Jagannathan (2012) and Grullon, Lyandres, and Zhdanov (2012), who argue that the poor empirical performance of the unconditional CAPM is mainly attributable to real options.

The paper is organized as follows. In the next section, we provide a brief summary of the related literature. Section II builds a simple example to develop the intuition. Section III presents the continuous-time model with investment and competition, and section IV discusses asset pricing implications. Section V presents simulation results and compares them with observed empirical regularities. The last section concludes.

I. Literature

Early work by Brennan (1973) and Bossaerts and Green (1989) models dividends as the sum of persistent idiosyncratic shocks and a single systematic shock. In particular, Bossaerts and Green (1989) derive two factor arbitrage pricing theory restrictions on dynamic equilibrium asset returns. They show that conditional expected returns depend inversely on the current stock price and apply this intuition to explain, in particular, the abnormally high January returns of small stocks. In this study, we focus on the role of idiosyncratic shocks as valuable

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conditioning information and aim to explain a broad range of asset pricing anomalies. More important, we show that the asset pricing implications of Bossaerts and Green (1989) are general and not restricted to their additive dividend process. In other words, the effects of idiosyncratic profit shocks are relevant for *almost every* firm in the economy.

Berk (1995) shows that anomalies related to firm size arise from differences in firms' unobservable discount rates. Instead, our results are driven by variation in firms' cash flows and are thus complementary to those in Berk (1995). In our model, firms with low cash flows have small idiosyncratic profit components and thus high systematic risk. Anomalies can then show up not only in market values, but also in fundamentals, such as dividends and earnings. Adding the feedback of discount rates to market valuation would strengthen our results.

Our study contributes to the rapidly growing literature that links the theory of investment under uncertainty to determinants of the cross-section of stock returns. Berk, Green, and Naik (1999) were among the first to link firm investment options to risk. Gomes, Kogan, and Zhang (2003) build a general equilibrium model with perfect competition that generates value and size effects in cross-section. In their model, the cross-sectional differences in firms' risk are driven by the importance of growth options relative to assets in place. In contrast, our analysis focuses on the dynamics of idiosyncratic shocks across stocks.

Carlson, Fisher, and Giammarino (2004) model a firm that can expand its business by investing in new projects. In their model, operating leverage makes assets in place more risky than growth options, giving rise to book-to-market and size anomalies. Zhang (2005) models costly investment reversibility and a countercyclical price of risk. Specifically, he shows that in bad times assets in place are riskier than growth options because they are difficult to reduce. This effect leads to an unconditional value premium because the price of risk is high in bad times. Cooper (2006) develops similar intuition in a model with lumpy investment and a constant price of risk. Kogan and Papanikolaou (2012) build an equilibrium model with two aggregate sources of risk that have different implications for growth options and assets in place. We find value, size, and idiosyncratic volatility effects even when firms have no operating leverage and no growth options, and when the price of the single source of risk is

constant.

To provide a risk-based explanation for momentum, Sagi and Seasholes (2007) model a firm as a portfolio of risk-free and risky assets. They show that a long position in the risk-free asset results in a positive return autocorrelation, as high returns on the risky asset increase the weight of the risky asset in a portfolio and therefore increase portfolio risk. In contrast, we model a risky zero-beta asset instead of a risk-free asset. The volatility embedded in the zero-beta asset changes the intuition of Sagi and Seasholes since returns can now originate from either systematic or idiosyncratic sources and can thus either increase or decrease future risk. Therefore, we do not attempt to explain the time-series properties of individual asset returns and mainly focus on cross-sectional implications.

Our study is also related to the literature examining the effect of product market competition on the value of real options and the optimal timing of option exercise. Our primary interest lies in identifying how competition affects the dynamics of conditional betas, however, rather than on the optimal timing of investment options exercises.[3] Aguerrevere (2009) and Bena and Garlappi (2012) also examine competition and its impact on systematic risk. The main difference is that in their studies, competition affects risk mainly by reducing the value of growth opportunities, while in ours it affects value and risk of all assets.

II. Idiosyncratic Shocks and Firm Risk

We now develop a simple example to highlight the main economic mechanism in the paper. Consider a firm with value $V(x_i, y)$ that depends on both idiosyncratic shock, x_i, and systematic shock, y. We assume that $V(x_i, y)$ is a continuous, twice differentiable function, and that higher values of idiosyncratic shock x_i indicate better states of the world, i.e., $V_x(x_i, y) > 0$. Beta is defined as the sensitivity of relative changes in value to relative changes in systematic

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[3]For example, Grenadier (2002) argues that competition erodes the value of real options, thereby reducing the advantage of waiting to invest. In contrast, Leahy (1993) and Caballero and Pindyck (1996) argue that, despite the fact that the option to wait is less valuable in a competitive environment, irreversible investment is still delayed because the upside profits are lowered by new firm entry. By considering a nonlinear production technology, Novy-Marx (2007) shows that firms in a competitive industry may delay irreversible investment even longer than suggested by a neoclassical framework.

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shock

$$\beta_i = \frac{V_y(x_i, y) y}{V(x_i, y)}. \tag{1}$$

Clearly, equity beta in general depends on idiosyncratic shock x_i. Differentiating this expression with respect to x_i, we show that beta is independent of idiosyncratic shocks only in the "knife-edge" case when

$$V(x_i, y) = \frac{V_x(x_i, y) V_y(x_i, y)}{V_{xy}(x_i, y)}. \tag{2}$$

A class of value functions that satisfies this partial differential equation is multiplicatively separable functions of the form

$$V(x_i, y) = f(x_i) g(y), \tag{3}$$

which have been used extensively in the previous literature. It is worth examining the criterion (2) with care. For the majority of value functions this condition will not be satisfied and betas will depend on the history of idiosyncratic shock realizations. This result implies that characteristics such as size, book-to-market, and volatility must be correlated with expected returns.

For example, consider a firm with the additive value function

$$V(x_i, y) = f(x_i) + g(y) - c, \tag{4}$$

where the first term captures the value derived from idiosyncratic shocks, the second term captures value from systematic shocks, and c is the firm's long or short position in the risk-free asset.

From (1), the firm market beta is given by

$$\beta_i = \frac{g_y(y) y}{V(x_i, y)}. \tag{5}$$

It is easy to see that β_i in this case is decreasing in the idiosyncratic shock since the numerator in expression (5) is independent of x_i and the denominator is increasing in x_i. This result implies that a positive idiosyncratic shock simultaneously increases firm market value and decreases beta, giving rise to value and size effects in the cross-section of stock returns.

To link our results to prior literature, the expression (5) could be rewritten as follows:

$$\beta_i = 1 - \frac{f(x_i)}{V(x_i, y)} + \frac{g_y(y) y - g(y)}{V(x_i, y)} + \frac{c}{V(x_i, y)}. \tag{6}$$

The first term in (6) is normalized to one. The second term is responsible for the size effect since a higher value of shock x_i will simultaneously lead to higher firm value and lower beta. The third term appears only if function $g(y)$ is nonlinear in the systematic shock y. Whether this term increases or decreases the overall firm beta depends on the concavity/convexity of function $g(y)$. For example, growth options linked to systematic profitability shocks induce convexity in the value function and therefore tend to increase systematic risk. In contrast, product market competition limits a firm's profits and induces concavity in the value function, thereby decreasing firm beta. The last term in (6) can represent operating/financial leverage $(c > 0)$ or cash savings $(c < 0)$. This term has received considerable attention in the previous literature (see, e.g., Carlson, Fisher, and Giammarino (2004) and Sagi and Seasholes (2007)).

III. The Model

This section lays out a model that extends the simple example to a dynamic setting and incorporates the effects of investment and product market competition. The model facilitates a comparison of our results with those of previous studies and enables us to evaluate the economic importance of asset pricing anomalies in simulated data. We deliberately do not model operating leverage or a time-varying price of risk since previous work has already shown that these features can help generate size and value premia.

A. Model Setup

Each firm in the economy generates profit

$$\Pi_i = x_i + \rho_i Y Q^{-\varepsilon}, \tag{7}$$

where x_i is the idiosyncratic demand shock (e.g., tastes for the differentiated firm's product), ρ_i is the firm's sensitivity to the systematic demand shock Y, Q is the mass of firms, and $1/\varepsilon$ is the positive price elasticity of demand. Time subscripts are omitted throughout. The

profitability shocks follow geometric Brownian motions in the risk-neutral measure

$$dx_i/x_i = \mu_x dt + \sigma_x dz_i, \qquad (8)$$

$$dY/Y = \mu_y dt + \sigma_y dz_y, \qquad (9)$$

where dz_i and dz_y are increments of uncorrelated standard Wiener processes, $E[dz_i dz_y] = 0$ for all i. The idiosyncratic shocks have identical drifts and volatilities and are uncorrelated across firms, $E[dz_i dz_j] = 0$ for $i \neq j$.

We model product market competition by allowing new firm entry conditional on states of the economy. New firms can enter the market by paying a fixed cost R. The value of systematic shock Y is common knowledge prior to entry, and we assume that all prospective entrants receive identical initial draws of idiosyncratic shocks x_i. All firms are risk-neutral and infinitesimally small.[4] Since for tractability purposes we do not model optimal firm exit, we assume that the equilibrium mass of firms Q decays over time with intensity δ,

$$dQ = -\delta Q dt. \qquad (10)$$

When there is no entry, the number of firms deterministically decreases as in (10). Hence, by denoting $y = YQ^{-\varepsilon}$ and using Ito's lemma, we can write the dynamics of the process y as

$$dy = \left(\mu_y + \varepsilon\delta\right) y dt + \sigma_y y dz_y, \qquad (11)$$

where the additional term in the drift, $\varepsilon\delta$, appears because the expected decline in the mass of firms, Q, leads to a higher growth of y.

Since all prospective entrants are identical, they enter at the same threshold, $\overline{y}$. Thus, new entry endogenously limits growth in the systematic component of profits, similar to the exogenous limits to growth found in, for example, Carlson, Fisher, and Giammarino (2004). As a result, the process (11) has a reflecting barrier at $\overline{y}$, formally defined as

$$dy = \begin{cases} 0, & \text{for } y = \overline{y} \text{ and } dz_y > 0 \\ \widehat{\mu}_y y dt + \sigma_y y dz_y, & \text{otherwise} \end{cases} \qquad (12)$$

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[4]This assumption allows us to treat firms as price-takers and to ignore the effect of firms' own output on the equilibrium price. Aguerrevere (2009) presents a more general model of competition, in which firms also take into account the effect of their own output on product price.

where $\widehat{\mu}_y = \mu_y + \varepsilon\delta$.[5]

We model investment options in reduced form. In addition to receiving continuous profits (7), each firm has an opportunity to irreversibly expand production or improve technology by paying a fixed cost. For tractability purposes, we assume that a firm can separately exercise growth options linked to idiosyncratic shocks (x-options) and to systematic shocks (y-options).[6] Specifically, by paying an investment cost I_x a firm can increase the idiosyncratic component of its cash flows x_i by a factor $1 + \gamma_x$, and by spending $\rho_i I_y$ it can increase the systematic component of cash flows $\rho_i y$ by a factor $1 + \gamma_y$. We make the exercise cost of the y-option proportional to ρ_i to ensure that the cost of exercising options scales up appropriately with the size of the firm's assets. The exercise cost of the systematic option is assumed to be sufficiently small relative to the cost of entry to ensure that options are optimally exercised prior to reaching the competition boundary. Investment is irreversible and indivisible, and, unlike Ai and Kiku (2012), we do not allow the investment cost to change with the state of the economy.

B. Firm Value

The value of the firm $V_i = V(x_i, y)$ is given by

$$V_i = E \int \Pi_i e^{-\widehat{r}t} dt, \tag{13}$$

where $\widehat{r} = r + \delta$ is the discount rate and Π_i is firm's instantaneous profit. The discount rate is adjusted to reflect the risks of exogenous exit of each firm. Firm value is obtained by solving the partial differential equation

$$\widehat{r}V_i = \Pi_i + \frac{\partial V_i}{\partial x_i}\mu_x x_i + \frac{\partial V_i}{\partial y}\widehat{\mu}_y y + \frac{1}{2}\frac{\partial^2 V_i}{\partial x_i^2}\sigma_x^2 x_i^2 + \frac{1}{2}\frac{\partial^2 V_i}{\partial y^2}\sigma_y^2 y^2. \tag{14}$$

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[5]As in Caballero and Pindyck (1996), the reflecting barrier $\overline{y}$ is time-invariant in our setting. A sufficient condition for this is a stationary distribution of the number of entering and exiting firms.

[6]Alternatively, growth options could be modeled to depend on both idiosyncratic and systematic profits, and increase *total* firm cash flows. Such an approach does not change the intuition developed in this paper, but creates significant complications due to the two-dimensional option exercise policy.

To prevent arbitrage in the model, we require the boundary condition that firm value is insensitive to changes in the systematic shock y as it approaches the reflecting barrier,

$$\left.\frac{\partial V_i}{\partial y}\right|_{y=\overline{y}} = 0. \tag{15}$$

In addition, firm value changes by the amount of external financing at the time growth options are exercised, and optimal option exercise requires the smooth-pasting conditions on the first derivatives to be satisfied (Dumas (1991) and Dixit (1993)).

The solution for firm value is summarized by the following proposition.

Proposition 1. *Denote by ι_x and ι_y the indicator functions, equal to one if the respective growth option has been exercised. Then the market value of the firm is given by*

$$V_i = V_i^{AX} + V_i^{GX} + \rho_i \left(V^{AY} + V^{GY} - V^{CY} \right), \tag{16}$$

where the value components $V_i^{AX}, V_i^{GX}, V^{AY}, V^{GY}$, and V^{CY} are given by

$$V_i^{AX}(x_i) = \frac{(1 + \iota_x \gamma_x) x_i}{\widehat{r} - \mu_x}, \tag{17}$$

$$V_i^{GX}(x_i) = \frac{(1 - \iota_x) \gamma_x x^*}{(\widehat{r} - \mu_x) d_2} \left(\frac{x_i}{x^*}\right)^{d_2}, \tag{18}$$

$$V^{AY}(y) = \frac{(1 + \iota_y \gamma_y) y}{\widehat{r} - \widehat{\mu}_y}, \tag{19}$$

$$V^{GY}(y) = \frac{(1 - \iota_y) \gamma_y y^*}{(\widehat{r} - \widehat{\mu}_y) b_2} \left(\frac{y}{y^*}\right)^{b_2}, \tag{20}$$

$$V^{CY}(y) = \frac{(1 + \gamma_y) \overline{y}}{(\widehat{r} - \widehat{\mu}_y) b_2} \left(\frac{y}{\overline{y}}\right)^{b_2}, \tag{21}$$

and the constants $b_2 > 1$ and $d_2 > 1$, the option exercise thresholds x^* and y^*, and the entry threshold $\overline{y} > y^*$ are given in the Appendix.

Proof of Proposition 1. See Appendix B.

Having derived firm market values, optimal investment strategies, and an entry threshold for new firms, we now turn to the analysis of systematic risk.

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C. Equity Betas

Since the systematic shock y represents aggregate uncertainty in the model, the firm's equity beta is the elasticity of the firm market value with respect to this shock.[7] In the following proposition, we derive factor betas.

Proposition 2. *Adopting the notation of Proposition 1, the factor beta of the firm is*

$$\beta_i = 1 - \frac{V_i^{AX}}{V_i} - \frac{V_i^{GX}}{V_i} + \rho_i (b_2 - 1) \left(\frac{V^{GY}}{V_i} - \frac{V^{CY}}{V_i} \right). \tag{22}$$

Proof of Proposition 2. See Appendix B.

The first term in (22) is normalized to one. The second term appears because part of firm value is derived from profits uncorrelated with aggregate demand uncertainty and reduces the overall firm's exposure to systematic risk. The third and fourth terms show that beta decreases because of growth options linked to idiosyncratic profitability shocks (x-options) and increases because of growth options linked to systematic shocks (y-options) since the latter are more sensitive to the underlying shocks than assets in place. Finally, the last term appears because of the limiting effect of competition on the systematic part of the firm's cash flows.

D. Book-to-Market and Price-Earnings Ratios

To relate firm characteristics to risk, we now specify the evolution for book values and earnings. We calculate book values based on the cost incurred per unit of installed capital. Since at the time of the option exercise, γ_x units of x-assets are added at cost I_x, and $\rho_i \gamma_y$ units of y-assets at cost $\rho_i I_y$, the initial book value is set to

$$B_i = \frac{I_x}{\gamma_x} + \rho_i \frac{I_y}{\gamma_y}. \tag{23}$$

Book value increases by I_x and $\rho_i I_y$ for corresponding idiosyncratic and systematic growth option exercises. Since firms in our model have no leverage, the book-to-market ratio is given by $\frac{B_i}{V_i}$.

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[7]Appendix C discusses the relation between factor betas with respect to the systematic shock y derived here and betas with respect to the aggregate market.

The price-earnings ratio PE_i is computed using the firm's profits (7) with an adjustment for asset expansion

$$PE_i = \frac{V_i}{(1 + \iota_x \gamma_x) x_i + (1 + \iota_y \gamma_y) \rho_i y}.$$ (24)

The price-earnings ratio is a function of both shocks x_i and y and therefore varies over time and in the cross-section.

IV. Asset Pricing Implications

Next we use Propositions 1 and 2 to evaluate the ability of the model to explain asset-pricing anomalies. To facilitate discussion, we first obtain the sensitivity of market values, firm betas, and price-earnings ratios to the idiosyncratic profitability shock x_i,

$$\frac{\partial V_i}{\partial x_i} = \frac{1}{x_i} \left(V_i^{AX} + d_2 V_i^{GX} \right) \quad > 0,$$ (25)

$$\frac{\partial \beta_i}{\partial x_i} = -\frac{\rho_i}{V_i^2} \frac{1}{x_i} \left(V_i^{AX} + d_2 V_i^{GX} \right) \left(V^{AY} + b_2 V^{GY} - b_2 V^{CY} \right) \quad < 0,$$ (26)

$$\frac{\partial PE_i}{\partial x_i} = \frac{\rho_i}{\Pi_i^2} \left((\mu_x - \widehat{\mu}_y) \frac{V_i^{AX} V^{AY}}{x_i} - V^{GY} + V^{CY} \right) + \frac{V_i^{GX}}{\Pi_i^2} \left(d_2 - 1 + d_2 \frac{\rho_i y}{x_i} \right).$$ (27)

Since a positive idiosyncratic shock represents good news and $d_2 > 0$, it is evident from (25) that firm market value is increasing in the idiosyncratic shock. Further, beta is decreasing in shock x_i since $y^* < \bar{y}$ and $b_2 > 0$. Therefore, it follows that a positive idiosyncratic shock simultaneously increases firm market value and decreases beta, leading to the size and book-to-market anomalies. We now discuss how different ingredients of the model affect the magnitudes of asset pricing anomalies.

A. The Benchmark Model

Consider first the benchmark model with no real options or competition ($V_i^{GX} = V^{GY} = V^{CY} = 0$). As outlined above, the negative relation between market capitalization of the firm and beta leads to the size and book-to-market effects. However, since there is no time-series variation in book values, the two anomalies are indistinguishable in this case. As suggested by equation (27), price-earnings ratios are constant in the benchmark model if the effective

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risk-neutral drifts are identical ($\mu_x = \widehat{\mu}_y$), and therefore they are unrelated to expected stock returns.

The benchmark model is also able to generate the negative relation between stock returns and idiosyncratic volatility. A positive idiosyncratic shock results in a larger idiosyncratic share of profits and hence higher idiosyncratic volatility of stock returns, while it also lowers systematic firm risk.

Return reversals arise naturally in the model and are caused by the evolution of idiosyncratic profitability shocks. This implies in particular that we cannot generate momentum from the dynamics of idiosyncratic shocks. Sagi and Seasholes (2007) show how systematic shocks in the presence of cash holdings can lead to momentum. Their intuition is nested in our model if idiosyncratic cash flow shocks have low volatility and the cross-sectional dispersion in exposure to the systematic shock, ρ_i, is large.

B. Growth Options

The general form of our profit function suggests a role for growth options that derive their value from the idiosyncratic profit component. We therefore analyze the effect of growth options on firm risk and their importance for asset pricing.

Since growth options can be viewed as levered claims on assets in place, the relation between the value of options and overall firm risk is typically positive.[8] We show that this intuition breaks down once we allow for options to depend on idiosyncratic cash flow shocks. Specifically, all growth options increase firm market value. In line with previous literature, a higher value of the systematic option increases the firm's exposure to systematic risk. However, larger idiosyncratic options imply a smaller overall beta, as shown in Proposition 2. Therefore, depending on their nature, growth options can lead to either lower or higher firm risk.

Prior empirical literature suggests that growth options are related to asset pricing anomalies. In particular, Da, Guo, and Jagannathan (2012) and Grullon, Lyandres, and Zhdanov (2012) show that the unconditional CAPM performs better in the absence of growth options.

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[8]Note that operating or financial leverage can change this relation, as pointed out by Zhang (2005), Carlson, Fisher, and Giammarino (2006), and Novy-Marx (2011)).

To see how growth options affect anomalies in our model, we again look at the sensitivity of firm values and betas to idiosyncratic profit shocks in equations (25) and (26).

Since b_2 and d_2 are greater than one, it is easy to see that with x-options firm market values and hence the book-to-market ratios become more sensitive to the idiosyncratic shocks. The systematic options have no effect on this sensitivity. At the same time, the sensitivity of betas to shocks x_i increases with both systematic and idiosyncratic options. In particular, it follows from (26) that, conditional on the same total firm value, a firm that derives more value from growth options will have a higher sensitivity of beta to the idiosyncratic profitability shocks. Overall, these results imply that growth options, particularly those linked to idiosyncratic shocks, magnify value and size effects in the model.

Finally, growth options add to our understanding of the predictive ability of price-earnings ratios. While constant in the benchmark model, price-earnings ratios fluctuate with idiosyncratic and systematic shocks if firms have expansion options. Since betas decrease with idiosyncratic shocks, equation (27) shows that the relation between price-earnings ratios and firm risk is in general ambiguous. The empirically observed negative relation obtains when idiosyncratic options are large compared to systematic ones.

C. Competition

We now analyze the effect of product market competition on cross-sectional anomalies. Equations (25) and (26) show that the sensitivity of firm market values to idiosyncratic shocks is independent of competition, while the sensitivity of betas is attenuated by competition. This implies that book-to-market and size effects weaken as a result of higher competition (lower $\bar{y}$). This prediction is consistent with empirical evidence in Hou and Robinson (2006), who document a larger book-to-market premium in more concentrated industries.

Finally, equation (27) shows that competition increases the sensitivity of price-earning ratios to idiosyncratic shocks and can strengthen the negative relation between price-earning ratios and firm risk.

D. Option Exercise

We now show that option exercise in our model predicts low future returns. This is consistent with the empirical evidence on underperformance following share issuances (Loughran and Ritter (1995), Daniel and Titman (2006), Pontiff and Woodgate (2008)) and the negative relation between asset growth and stock returns in the cross-section (Cooper, Gulen, and Schill (2008)). The following proposition shows that any option exercise (either x or y-type) leads to a decline in equity beta, provided that the new investment is financed by equity issuance.

Proposition 3. *The market beta of the firm declines at the exercise of the idiosyncratic or systematic growth options if investment is financed by new equity issuance.*

Proof of Proposition 3. See Appendix B.

As in Carlson, Fisher, and Giammarino (2006), systematic risk decreases following the exercise of options linked to systematic shocks because such options are more sensitive to the priced factor than are assets in place. The proposition shows that the risk also decreases at exercise of the idiosyncratic option, albeit for a different reason. Since the firm raises external financing and uses it to invest in idiosyncratic assets, systematic profits become a smaller fraction of total firm value, reducing beta. If the firm finances investment by taking on new debt or with its own cash reserves, then firm betas do not change at the time of growth option exercises. Financial leverage increases with debt issuance or the reduction of cash holdings, and this effect exactly offsets the reduction in leverage from exchanging options into assets in place.

V. Simulation Results

In this section, we use simulations to evaluate the ability of our framework to reproduce the key features of stock return data. Since closed form solutions to the model are available, we use them directly to generate a panel of firms over time. We first discuss calibrations of the model parameters and then examine the properties of the generated data.

A. Calibration

Table I summarizes the parameters used in the calibration. We simulate monthly data for $N = 100$ economies and $n = 2,000$ firms over 45 years. The first five years are discarded to ensure sufficient variation in firm characteristics in the cross-section, resulting in time-series of $T = 40$ years, roughly in line with data used in previous empirical studies (e.g., Fama and French (1992)). Both profitability shocks have mean growth rates of $\mu_y = \mu_x = 0.03$, corresponding to annual earnings growth. Volatilities are set to $\sigma_y = 0.15$ and $\sigma_x = 0.25$. The risk-free rate is $r_f = 0.04$, and the price of risk associated with the y-factor is $\lambda = 0.15$. Under our parametrization, this price of risk implies an average equity risk premium of approximately 4% per year. The price elasticity of demand is set to $1/\varepsilon = 1/0.5 = 2$, similar to an elasticity of 1.6 used in Aguerrevere (2009).

The remaining parameters are chosen as follows: the number of firms decays at an annual rate of $\delta = 0.02$. Firms' exposure to the systematic shock, ρ_i, is uniformly distributed on the interval $[0.5, 1.5]$. Exercises of systematic and idiosyncratic growth options double the corresponding cash flows, $\gamma_x = \gamma_y = 1$. The initial values as well as exercise and entry costs are shown in Table I. They are selected such that firms exercise their options before competition suppresses further growth, and about 70% of all idiosyncratic and systematic options are exercised over the sample period. Across economies, entry cost ensures that the competition boundary is reached in approximately half of the simulated economies.

We construct realized returns as follows. First, we simulate the model forward to obtain the full history of firm dividends and values given the initial conditions and realizations of the shocks $x_{i,t}$ and y_t. Second, using these values we compute the realized holding period returns in the risk-neutral measure. Finally, to these returns we add the risk premium estimated as the individual firm's beta multiplied by the per-period risk premium. The following formula summarizes

$$r_{i,t} = \frac{V_{i,t} + \Pi_{i,t} - V_{i,t-1} - I_x \iota_x - \rho_i I_y \iota_y}{V_{i,t-1}} + \lambda \beta_{i,t}, \tag{28}$$

where $\Pi_{i,t}$ denotes the dividend process, and the returns are adjusted for external financing of investments.

B. Results

Figures 1 and 2 illustrate the dynamics of the main variables in the model. Specifically, in Panel A of Figure 1 we display a sample path of the systematic component of cash flows, y. Note that y bounces back whenever it reaches the upper reflective barrier, $\overline{y}$, where more firms enter the market.

In Panel B of Figure 1, we plot the evolution of the mass of firms, Q, corresponding to the path of y. The smooth downward adjustment in the mass of firms is due to gradual decay in the number of firms, while the upward jumps are caused by entry of new firms. Note that firms tend to enter the market following favorable systematic shocks, and more firms tend to enter when there are fewer competitors in the market.

Panel A of Figure 2 displays three sample paths of x-shocks from the simulated economy in Figure 1. The vertical lines indicate when the idiosyncratic investment options are exercised. One of the three firms does not exercise its idiosyncratic option during the observation period.

We next compute firm values at each point in time and plot them in Panel B. Firms with higher idiosyncratic shocks exercise their options sooner. Observe also that firm values jump at the point of option exercises. This is caused by an inflow of external funds to finance firm expansion.

The book-to-market ratios in Panel C fluctuate as x and y-shocks evolve. Additionally, book-to-market ratios change discontinuously at the time investment options are exercised. At exercise, both market value and book value increase by the same amount, the cost of investment. Since book-to-market ratios in our model are typically below one, exercise usually increases the book-to-market ratio.

The price-earnings ratios displayed in Panel D fluctuate with profitability shocks because the ratio of option value to assets in place changes. Price-earnings ratios drop sharply at option exercises because options are replaced with newly installed assets in place. Further, observe that the limit to growth in the y-component depresses price-earnings ratios for all firms in the economy. This is particularly evident when y approaches the competition threshold.

Panel E illustrates that factor betas can change over time in response to several effects.

First, given a systematic shock, firms' betas decrease with idiosyncratic shocks. This effect drives most of the gradual changes in beta in the graph. Second, when systematic investment options (y-options) are exercised and converted into assets in place, there is an instant drop in beta because assets in place have a lower sensitivity to the value of the shock. As we have argued theoretically, betas also decline at the exercise of idiosyncratic options (x-options) because the part of firm value which is unrelated to market risk increases by the amount of new equity financing. Third, factor betas decline to zero in the proximity of the competition threshold.

In Panel F, we display the corresponding market betas. Construction of market betas relies on the assumption that the market value is the sum of all firm values in the economy. Market betas are the ratio of individual factor betas and the sensitivity of the market to the factor. Details are described in Appendix C. By construction, the average market beta is one, and in particular does not decline to zero in times of strong competition as the average factor beta does. The cross-sectional dispersion in market betas is higher before options are exercised, and when the systematic shock is smaller.

We now conduct asset pricing tests on the simulated data. Every year, using the simulated panel of data, we form 20 portfolios based on ranked book-to-market ratios, market capitalizations, price-earnings ratios, and idiosyncratic volatilities at the beginning of the year. We weight stocks equally within each portfolio, and hold the portfolios for twelve months. Time-series average returns are calculated for each portfolio.

Panel A of Figure 3 provides a scatter plot of the relation between average returns of the 20 book-to-market portfolios and average log book-to-market ratios. The relation between book-to-market ratios and returns is positive and nearly linear. Similarly, Panel B documents that the average realized returns monotonically decrease in firm size.

Returns of price-earnings portfolios shown in Panel C exhibit a non-monotonic pattern. In our calibration, the exogenous decay in the number of firms implies a larger effective growth rate for systematic shocks, and therefore higher valuation ratios associated with y-earnings. As a result, in the absence of growth options and competition, price-earnings ratios

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are positively related to returns. Growth options and competition reverse this relation, which is clearly visible in the tails of the graph.

Returns of portfolios sorted on idiosyncratic volatility are shown in Panel D. Idiosyncratic volatility is estimated as the residual standard deviation from time-series regressions of stock returns onto changes in the systematic profitability shock over the 24 months prior to portfolio formation. Consistent with empirical evidence, high idiosyncratic risk is associated with low returns.

Our evidence suggests that the model has the potential to explain four common asset pricing anomalies in a univariate setting. We now turn to cross-sectional Fama-MacBeth regressions to evaluate our model's multivariate performance. Table II reports average Fama-MacBeth coefficients across 100 simulated economies and the corresponding average t-statistics for each coefficient. In each month t, realized stock returns are regressed on theoretical betas (β_t), estimated betas $(\widehat{\beta}_t)$, the log book-to-market ratio (B/M), log firm value $(Size)$, the prior 12-month returns (MOM), log price-earnings ratio (P/E), and the log of idiosyncratic volatility $(IVol)$. Betas and idiosyncratic volatility are estimated, respectively, as slope coefficient and residual standard deviation from time-series regressions of stock returns onto changes in the systematic profitability shock from month $t-24$ to $t-1$.

As a reference, the first row in the table shows that theoretical conditional betas are, not surprisingly, highly significant and the factor risk premium is 1.16% per month. Specification II replaces true betas with their estimated counterparts. While empirical betas are also strongly related to returns, the estimated risk premium drops because of measurement error in the explanatory variable. Specifications III and IV demonstrate that there is a significant positive relation between the realized returns and book-to-market ratios in the simulated data, and a negative relation between stock returns and firm size. Regression V shows that, under our parametrization, the model cannot explain momentum. Price-earnings ratios and idiosyncratic volatility in specifications VI and VII are negatively related to returns. Specification VIII shows that, when beta is estimated, both size and book-to-market ratio are significant return predictors. Since our model is a conditional one-factor model, true theoretical betas

in regression IX drive out all other variables.

In Table III, we document the magnitudes of the cross-sectional effects generated by the model across decile portfolios. For example, the difference in average stock returns between the top and bottom deciles by book-to-market ratio amounts to 56 basis points per month. This premium is fully explained by the difference in betas of 0.45. Similarly, returns of size-sorted portfolios differ by 54, price-earnings portfolios by 17, and idiosyncratic volatility portfolios by 56 basis points.

We next analyze how the particular model components contribute to the magnitude of return differences across portfolios. Specifically, we sort the model-generated data into characteristic deciles for the following model specifications: the full model (Panel A), the model without growth options ($\gamma_x = \gamma_y = 0$, Panel B), and the model without competition ($R \to \infty$, Panel C). Table IV shows the results. As we have argued, removing growth options tends to decrease the magnitude of the book-to-market, size, and idiosyncratic volatility premia. In contrast, shutting down competition increases the premia. While the model without competition generates a positive relation between price-earnings ratios and returns, the full model is able to match the empirically observed negative relation.

Table V shows that options are necessary to the generate predictive ability of both book-to-market and size in multivariate regressions. In particular, in the model without options, the book-to-market ratio significantly predicts returns, but size does not. Size in our model is not a sufficient statistic for beta because firms can be larger because of either a high idiosyncratic shock x_i or a large loading ρ_i on the systematic factor. At option exercise, the size effect is reinforced relative to the value effect since both the market capitalization and the book-to-market ratio increase and beta declines. The table also shows that competition reduces the magnitudes of both effects.

In Figure 4, we plot key firm characteristics before and after growth option exercises, averaged across firms and economies. Panels A and B show the dynamics of firm characteristics during the 48-month period centered on the exercise of y and x-options, respectively. The two plots for market values show that options are typically exercised following high stock returns.

Values jump at exercise due to the injection of additional cash in the firm. Importantly, the market capitalization increases with the issuance of new shares. There are no arbitrage opportunities because the price per share is unchanged. In contrast, book-to-market ratios tend to decrease prior to the option exercise. At the time of investment, both the book and market values increase by the same amounts. Since the average book-to-market ratio is below one, the ratio typically increases at exercise.

Finally, we plot conditional factor betas around the exercise of systematic and idiosyncratic options. As we have argued theoretically, betas decrease at the time of x and y-option exercises under external equity financing. Exercising y-options lowers the sensitivity to the systematic profitability shock since assets in place are less risky than growth options. In contrast, exercising x-options increases the value of assets derived from the idiosyncratic component, thereby lowering firm risk. The distinctive pattern in the dynamics of pre-exercise betas is informative. Consistent with prior literature (e.g., Carlson, Fisher, and Giammarino (2010)), the average beta tends to increase prior to the y-option exercise. The increase in both the systematic shock and the value of the option prior to exercise leads to a rise in beta. Conversely, the average beta decreases right before the x-option is exercised since this option depends on the idiosyncratic profit component.

To better understand the conditional nature of betas and the value premium, we plot in Figure 5 the market risk premium, measures of cross-sectional variation in factor and market betas, and the realized value premium against the level of the systematic shock y. The plots on the left (Panel A) show results in an economy without competition; Panel B shows results for the full model with competition. The first row plots the relation between the market risk premium and the factor y. The market risk premium is defined as the product of the weighted average factor beta, β_M^y, and the constant price of y risk, λ. As expected, β_M^y and consequently the market risk premium is monotonically increasing in y in the absence of competition. A higher value of y implies that a larger part of total market value is systematic, leading to higher systematic risk. Introducing competition breaks this link. While the average firm beta initially increases in y, it is non-monotonic. As predicted by Proposition 2, firm

betas decrease as y approaches the competition boundary. With the assumed factor risk premium of 15% annually, the model-implied market risk premium ranges from 0 to about 5%.

The magnitudes of asset pricing anomalies in our model are related to the cross-sectional variation in firm betas. We measure this variation as the difference between the average betas of the first and tenth decile of beta-sorted portfolios. Variation in factor and market betas is shown in rows two and three, respectively. The spread in factor betas, $\beta_{10}^y - \beta_1^y$, is highest for intermediate values of y. If the systematic shock is small relative to typical idiosyncratic shocks, most betas are close to zero, which in turn leads to low cross-sectional variation. As y gets very large, in the case without competition, betas tend to move toward one, again reducing variation. The results for the case with competition are similar but more pronounced, and the economic reasoning is different. As y approaches the competition boundary, new entry makes firms less sensitive to systematic shocks, thereby decreasing all factor betas to zero. In general, the effect of the option exercise on risk is larger for high beta than low beta firms, leading to a decrease in the cross-sectional dispersion of risk. The slope in the plots is steeper in the regions of y where option exercise is common, and it flattens once all firms have expanded.

The spreads in market betas in the third row show a different pattern. Since for a small systematic shock all individual factor betas are close to zero, rescaling results in a large cross-sectional dispersion in market betas. The spread then decreases as y increases.

Finally, the last row displays the realized value premium,, defined as the annual return difference between the top and bottom decile of book-to-market sorted portfolios. The realized value premium initially increases in y and later decreases, closely mimicking the behavior of the factor beta spread $\beta_{10}^y - \beta_1^y$. This suggests that the value premium in our model is highest in bad times.

VI. Conclusion

Cross-sectional anomalies in stock returns have long presented a challenge to standard asset pricing models. We show that, under general assumptions, firms' conditional betas directly depend on the history of idiosyncratic shocks and vary over time. Firm value is negatively related to risk because positive idiosyncratic shocks to cash flows increase market capitalization and simultaneously lead to a decrease in systematic risk. This size effect is distinct from the feedback of discount rates into market values (Berk (1995)), and holds for both market-based and accounting-based measures of firm size. Similarly, the model is able to replicate the value anomaly and the empirically observed negative relation between idiosyncratic volatility and expected returns.

We show that real options magnify the anomalies in the model, allow for separation of value and size effects, and generate return predictability by price-earnings ratios. Since growth options can depend on systematic or idiosyncratic profits, they can either increase or decrease the firm's factor risk. Product market competition tends to attenuate the magnitudes of cross-sectional anomalies. The analysis of the data generated by the model confirms that the model can produce reasonable magnitudes of value, size, price-earnings, and idiosyncratic volatility anomalies both in portfolio sorts and in Fama and MacBeth (1973) cross-sectional regressions. Overall, our results imply that any economic variable correlated with the history of idiosyncratic cash flow shocks can help to explain expected stock returns.

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Appendix

A. Notational Key

Q	Equilibrium mass of firms in the economy
δ	Decay intensity in mass of firms Q
$1/\varepsilon$	Price elasticity of demand
Y	Systematic profitability shock
ρ_i	Profit function constant
x_i, y	Idiosyncratic and systematic cash flow components
x^*, y^*	Exercise thresholds for x-option and y-option
$I_x, \rho_i I_y$	Cost of exercising the respective options
ι_x, ι_y	Indicators for exercising the options
γ_x, γ_y	Investment scale
R	Cost of entry
$\overline{y}$	Limit to growth in y from firm competition
V_i	Market value of firm i
V_i^{AX}	Market value of assets in place associated with idiosyncratic profitability
V_i^{GX}	Market value of growth options associated with idiosyncratic profitability
V^{AY}	Market value of assets in place associated with systematic profitability
V^{GY}	Market value of growth options associated with systematic profitability
V^{CY}	Loss in market value due to competition

B. Proofs

Proof of Proposition 1. Consider a trial solution

$$V\left(x_i, y\right) = v_1\left(x_i\right) + v_2\left(y\right). \tag{29}$$

Since equation (14) is additively separable in variables x_i and y, the general solution to (14) is equal to the sum of the ODE solution for $v_1\left(x_i\right)$ and the ODE solution for $v_2(y)$. Consider the "continuation" problem of the firm that exercised all its options. Its value is given by

$$\widehat{V}\left(x_i, y\right) = \frac{\left(1+\gamma_x\right)x_i}{\widehat{r} - \mu_x} + \frac{\left(1+\gamma_y\right)\rho_i y}{\widehat{r} - \widehat{\mu}_y} + Ay^{b_2}, \tag{30}$$

where the last term is negative and appears because of the limiting effect of competition, and b_2 is the positive root of the quadratic equation

$$b^2\sigma_y^2 + b\left(2\widehat{\mu}_y - \sigma_y^2\right) - 2\widehat{r} = 0. \tag{31}$$

Prior to the exercise of the option, the general solution for firm value is given by

$$V\left(x_i, y\right) = \frac{x_i}{\widehat{r} - \mu_x} + \frac{\rho_i y}{\widehat{r} - \widehat{\mu}_y} + By^{b_2} + Cx_i^{d_2}, \tag{32}$$

where B and C are constants, b_2 is the positive root of (31), and d_2 is the positive root of a similar equation for x

$$d^2\sigma_x^2 + d\left(2\mu_x - \sigma_x^2\right) - 2\widehat{r} = 0. \tag{33}$$

At the time of the exercise, the value of the firm is equal to the value after the exercise minus the investment cost (the value-matching conditions)

$$V\left(x^*, y\right) = \widehat{V}\left(x^*, y\right) - I_x, \tag{34}$$

$$V\left(x_i, y^*\right) = \widehat{V}\left(x_i, y^*\right) - \rho_i I_y, \tag{35}$$

where $\widehat{V}$ is given by (30). Note that since firm value is separable in the x and y components, the exercise of one option does not affect the exercise policy for another. For the exercise to be optimal, an additional condition known as smooth-pasting or high-contact condition (Dumas (1991), Dixit (1993)) has to be satisfied,

$$V_x\left(x^*, y\right) = \widehat{V}_x\left(x^*, y\right), \tag{36}$$

$$V_y\left(x_i, y^*\right) = \widehat{V}_y\left(x_i, y^*\right). \tag{37}$$

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Using (15) and (34)-(37), we find constants A, B, and C and well as pre- and post-exercise firm values,

$$\widehat{V}(x_i, y) = \frac{(1 + \gamma_x) x_i}{\widehat{r} - \mu_x} + \frac{(1 + \gamma_y) \rho_i y}{\widehat{r} - \widehat{\mu}_y} - \frac{(1 + \gamma_y) \rho_i \overline{y}}{(\widehat{r} - \widehat{\mu}_y) b_2} \left(\frac{y}{\overline{y}}\right)^{b_2}, \tag{38}$$

and

$$\begin{aligned}
V(x_i, y) &= \frac{(1 + \iota_x \gamma_x) x_i}{\widehat{r} - \mu_x} + \frac{(1 - \iota_x) \gamma_x x^*}{(\widehat{r} - \mu_x) d_2} \left(\frac{x_i}{x^*}\right)^{d_2} \\
&\quad + \frac{(1 + \gamma_y \iota_y) \rho_i y}{\widehat{r} - \widehat{\mu}_y} + \frac{(1 - \iota_y) \gamma_y \rho_i y^*}{(\widehat{r} - \widehat{\mu}_y) b_2} \left(\frac{y}{y^*}\right)^{b_2} - \frac{(1 + \gamma_y) \rho_i \overline{y}}{(\widehat{r} - \widehat{\mu}_y) b_2} \left(\frac{y}{\overline{y}}\right)^{b_2}. \tag{39}
\end{aligned}$$

The thresholds for exercise x^* and y^* are then defined as

$$x^* = \frac{d_2}{d_2 - 1} \frac{(\widehat{r} - \mu_x) I_x}{\gamma_x}, \tag{40}$$

$$y^* = \frac{b_2}{b_2 - 1} \frac{(\widehat{r} - \widehat{\mu}_y) I_y}{\gamma_y}. \tag{41}$$

Note that y^* is identical for all firms since both investment benefits and costs are proportional to ρ_i. Since entry into the market is competitive, we follow Leahy (1993) and Caballero and Pindyck (1996) to require that expected profit at entry be zero,

$$V(x_0, \overline{y}) = R, \tag{42}$$

where x_0 is the expected idiosyncratic shock x_i and R is the cost of entry. We assume that the y-options are exercised prior to reaching the reflecting barrier. Since new firms enter at $\overline{y} > y^*$, they have no y-options but have investment options linked to the idiosyncratic profit component. Thus we can solve for the limit to growth parameter $\overline{y}$ from (42), which yields

$$\overline{y} = \left(R - \frac{x_0}{\widehat{r} - \mu_x} - \frac{\gamma_x x^*}{(\widehat{r} - \mu_x) d_2} \left(\frac{x_0}{x^*}\right)^{d_2}\right) \frac{b_2}{b_2 - 1} \frac{\widehat{r} - \widehat{\mu}_y}{(1 + \gamma_y) \rho_0}, \tag{43}$$

where ρ_0 is the expected sensitivity to the systematic shock. It is clear from the equation above that the entry threshold $\overline{y}$ is time-independent and increases with the cost of entry R. $\qquad\square$

Proof of Proposition 2. Using the definition $\beta_i \equiv \frac{\partial V_i / V_i}{\partial y / y}$, the proof follows from differentiation of (16). $\qquad\square$

Proof of Proposition 3. We start by considering the effect of exercise of the option linked to idiosyncratic shocks. Using (22), the claim follows by taking the difference in betas just *after* and just *prior* to the exercise of the x-option.

$$\beta_i(x_+^*) - \beta_i(x_-^*) = \frac{-I_x\rho_i\left(V^{AY} + b_2 V^{GY} - b_2 V^{CY}\right)}{V\left(x^*, y\right)\left(V\left(x^*, y\right) + I_x\right)} < 0. \tag{44}$$

Since the expression in parentheses in the numerator is always positive when $y^* < \overline{y}$, we have the result that beta always decreases after the exercise of the x-option.

Similarly, for y-option exercise, the difference post-exercise and pre-exercise betas is

$$\beta_i(y_+^*) - \beta_i(y_-^*) = \frac{-\rho_i I_y\left(1 + \gamma_y\right)}{V\left(x_i, y^*\right)\left(V\left(x_i, y^*\right) + \rho_i I_y\right)}\frac{\rho_i y^*}{\widehat{r} - \widehat{\mu}_y}\left(1 - \left(\frac{y^*}{\overline{y}}\right)^{b_2 - 1}\right) < 0. \tag{45}$$

Again since $b_2 > 1$ and $y^* < \overline{y}$, we see that beta declines at the exercise of the y-option. $\quad\square$

C. Relation between Factor and Market Betas

The analysis in the text is based on betas with respect to the common risk factor y,

$$\beta_i^y = \frac{\partial V_i / V_i}{\partial y / y}. \tag{46}$$

We now connect our analysis to the commonly used "market betas" in the context of the CAPM.

Starting from a linear projection of stock returns on the priced factor,

$$R_i = \alpha_i + \beta_i^y y + \varepsilon, \tag{47}$$

it is straightforward to show that

$$\beta_i^M = \beta_i^y \beta_y^M, \tag{48}$$

where

$$\beta_y^M = \frac{Cov(y, R_m)}{Var(R_m)}. \tag{49}$$

We now aggregate individual firm values to obtain the value of the stock market and compute market betas. Since the market is the sum of values of N stocks,

$$M \equiv \Sigma_i V_i = \Sigma_i \left(V_i^{AX} + V_i^{GX} \right) + \left(V^{AY} + V^{GY} - V^{CY} \right) \Sigma_i \rho_i, \tag{50}$$

the beta of y with respect to the market is $\beta_y^M = \frac{\partial y / y}{\partial M / M}$ and can be found using implicit differentiation for $\frac{\partial y}{\partial M}$:

$$\beta_y^M = \frac{M}{\left(V^{AY} + b_2 V^{GY} - b_2 V^{CY} \right) \Sigma_i \rho_i}. \tag{51}$$

From Proposition 2, beta simplifies to

$$\beta_i^y = \frac{\rho_i}{V_i} \left(V^{AY} + b_2 V^{GY} - b_2 V^{CY} \right), \tag{52}$$

therefore, the equilibrium relation to the CAPM beta can be written as

$$\beta_i^M = \beta_i^y \beta_y^M = \frac{\Sigma_i V_i}{V_i} \frac{\rho_i}{\Sigma_i \rho_i}, \tag{53}$$

In particular, when all firms have the same sensitivity to the systematic shock, we have

$$\beta_i^M = \frac{\overline{V}}{V_i}, \tag{54}$$

where $\overline{V}$ is the average value of firms in the economy. Then, firms smaller than average will have betas above one, and firms larger than average will have betas below one. By construction, the weighted sum of market betas is equal to one, $\Sigma_i V_i \beta_i^M / \Sigma_i V_i = 1$.

Table I
Parameter Values Used in Simulations

This table lists the parameters used for the simulations.

Parameter	Notation	Value
Initial mass of firms	Q_0	1.5
Decay intensity for mass of firms	δ	0.02
Price elasticity of demand	$1/\varepsilon$	1/0.5
Distribution of sensitivity	ρ_i	$U\,[0.5, 1.5]$
Initial profitability shocks	(X_0, Y_0)	$(1, 1)$
Cost of entry	R	200
Cost of exercising options	$(I_x, \rho_i I_y)$	$(20, \rho_i 20)$
Volatility of profitability shocks	(σ_x, σ_y)	$(0.25, 0.15)$
Drift of profitability shocks	(μ_x, μ_y)	$(0.03, 0.03)$
Investment scale	(γ_x, γ_y)	$(1, 1)$
Simulation horizon (in years)	T	40
Number of simulated firms	n	$2,000$
Number of simulated economies	N	100
Price of risk for y-factor	λ	0.15
Risk-free rate	r_f	0.04

Table II
Fama-MacBeth Regressions on the Simulated Data

This table reports average Fama−MacBeth coefficients and average t-statistics from 100 simulations of a cross-section of 2,000 stocks over 45 years, the first 5 of which are discarded. In each month t, the realized stock returns are regressed on theoretical betas (β_t), estimated betas ($\widehat{\beta}_t$), the log book-to-market ratio (B/M), log firm value ($Size$), the prior 12-month returns (MOM), the log price-earnings ratio (P/E), and the log of idiosyncratic volatility ($IVol$). Betas and idiosyncratic volatility are estimated, respectively, as slope coefficient and residual standard deviation from time-series regressions of stock returns onto changes in the systematic profitability shock from month $t-24$ to $t-1$.

	β_t	$\widehat{\beta}_t$	B/M	$Size$	MOM	P/E	$IVol$
I	1.16						
	(4.96)						
II		0.35					
		(4.60)					
III			0.23				
			(5.76)				
IV				-0.18			
				(-5.91)			
V					-0.04		
					(-0.52)		
VI						-0.36	
						(-3.23)	
VII							-0.24
							(-5.79)
VIII		0.13	0.11	-0.08			
		(3.81)	(3.71)	(-4.07)			
IX	1.16		-0.00	-0.00			
	(4.50)		(-0.03)	(-0.07)			

Table III

Fama-French Portfolio Sorts on Simulated Data

This table reports average returns, estimated unconditional portfolio betas, and characteristics of 10 portfolios formed by book-to-market ratio (B/M, Panel A), size ($Size$, B), price-earnings ratio (P/E, C), and idiosyncratic volatility ($IVol$, D). IVol is estimated as residual standard deviation from time-series regressions of stock returns onto changes in the systematic profitability shock from month $t-24$ to $t-1$. The data consist of 100 simulations of a cross-section of 2,000 stocks over 45 years, the first 5 of which are discarded.

	Low	2	3	4	5	6	7	8	9	High	H–L
					Panel A: Book-to-Market Portfolios						
Ret	0.61	0.70	0.78	0.84	0.90	0.95	1.01	1.06	1.11	1.17	0.56
$\widehat{\beta}$	0.10	0.17	0.23	0.29	0.33	0.38	0.42	0.45	0.49	0.54	0.45
B/M	-2.16	-1.47	-1.20	-1.02	-0.88	-0.75	-0.63	-0.51	-0.38	-0.17	1.99
					Panel B: Size Portfolios						
Ret	1.16	1.09	1.06	1.02	0.97	0.91	0.84	0.77	0.70	0.62	-0.54
$\widehat{\beta}$	0.53	0.48	0.45	0.42	0.39	0.34	0.29	0.23	0.17	0.10	-0.43
$Size$	3.95	4.27	4.47	4.64	4.80	4.97	5.17	5.41	5.74	6.46	2.51
					Panel C: Price-Earnings Portfolios						
Ret	1.00	0.92	0.88	0.88	0.89	0.92	0.94	0.94	0.92	0.83	-0.17
$\widehat{\beta}$	0.43	0.35	0.32	0.31	0.32	0.33	0.35	0.37	0.35	0.27	-0.16
P/E	3.25	3.28	3.31	3.34	3.37	3.42	3.46	3.51	3.56	3.63	0.38
					Panel D: Idiosyncratic Volatility Portfolios						
Ret	1.21	1.13	1.06	0.99	0.93	0.87	0.81	0.76	0.71	0.65	-0.56
$\widehat{\beta}$	0.57	0.51	0.46	0.41	0.36	0.31	0.26	0.22	0.17	0.12	-0.45
$IVol$	-5.02	-4.36	-4.00	-3.73	-3.52	-3.34	-3.19	-3.06	-2.93	-2.76	2.26

Table IV
Returns of Fama-French Portfolio Sorts on Simulated Data

This table reports average monthly returns, betas, and characteristics of 10 portfolios formed by book-to-market (B/M), size ($Size$), price-earnings ratio (P/E), and idiosyncratic volatility ($IVol$) from our simulated data. IVol is estimated as residual standard deviation from time-series regressions of stock returns onto changes in the systematic profitability shock from month $t-24$ to $t-1$. The data consists of 100 simulations of a cross-section of 2,000 stocks over 45 years, the first 5 of which are discarded. Panel A contains results for our benchmark model. In Panels B and C, we present results for the model without growth options and competition, respectively.

	Low	2	3	4	5	6	7	8	9	High	H–L
					Panel A: Full model						
B/M	0.61	0.70	0.78	0.84	0.90	0.95	1.01	1.06	1.11	1.17	0.56
$Size$	1.16	1.09	1.06	1.02	0.97	0.91	0.84	0.77	0.70	0.62	-0.54
P/E	1.00	0.92	0.88	0.88	0.89	0.92	0.94	0.94	0.92	0.83	-0.17
$IVol$	1.21	1.13	1.06	0.99	0.93	0.87	0.81	0.76	0.71	0.65	-0.56
					Panel B: Model without options						
B/M	0.61	0.70	0.76	0.82	0.87	0.92	0.96	1.00	1.05	1.10	0.48
$Size$	1.07	1.02	0.99	0.97	0.93	0.89	0.83	0.77	0.71	0.62	-0.45
P/E	1.11	1.05	1.01	0.96	0.91	0.86	0.81	0.76	0.70	0.62	-0.49
$IVol$	1.14	1.06	1.00	0.95	0.90	0.85	0.80	0.75	0.71	0.64	-0.50
					Panel C: Model without competition						
B/M	0.90	1.14	1.25	1.33	1.39	1.44	1.49	1.53	1.57	1.63	0.73
$Size$	1.63	1.56	1.52	1.50	1.46	1.42	1.34	1.24	1.10	0.88	-0.75
P/E	0.85	1.05	1.21	1.35	1.47	1.52	1.51	1.52	1.56	1.61	0.76
$IVol$	1.71	1.64	1.58	1.51	1.44	1.36	1.27	1.18	1.06	0.89	-0.82

Table V
**Separating Value and Size**

This table reports average Fama−MacBeth coefficients and average t-statistics from 100 simulations of a cross-section of 2,000 stocks over 45 years, the first 5 of which are discarded. In each month t, the realized stock returns are regressed on estimated betas, log book-to-market ratios, and log firm values. Betas are estimated from time-series regressions of stock returns onto changes in the systematic profitability shock from month $t - 24$ to $t - 1$.

	$\widehat{\beta}_t$	B/M	$Size$
Full model	0.13	0.11	-0.08
	(3.81)	(3.71)	(-4.07)
Model w/o options	0.12	0.20	0.00
	(3.34)	(4.77)	(0.03)
Model w/o competition	0.48	0.12	-0.12
	(5.20)	(3.76)	(-4.97)

Figure 1. Simulated Sample Economy

This figure shows one sample path for the systematic profitability shock y (Panel A) and the respective dynamics of the mass of firms in the competitive economy (Panel B). The horizontal dashed line indicates the reflection barrier due to competition. The parameters are described in the text and summarized in Table I.

Panel A: Sample Path of Systematic Shock y
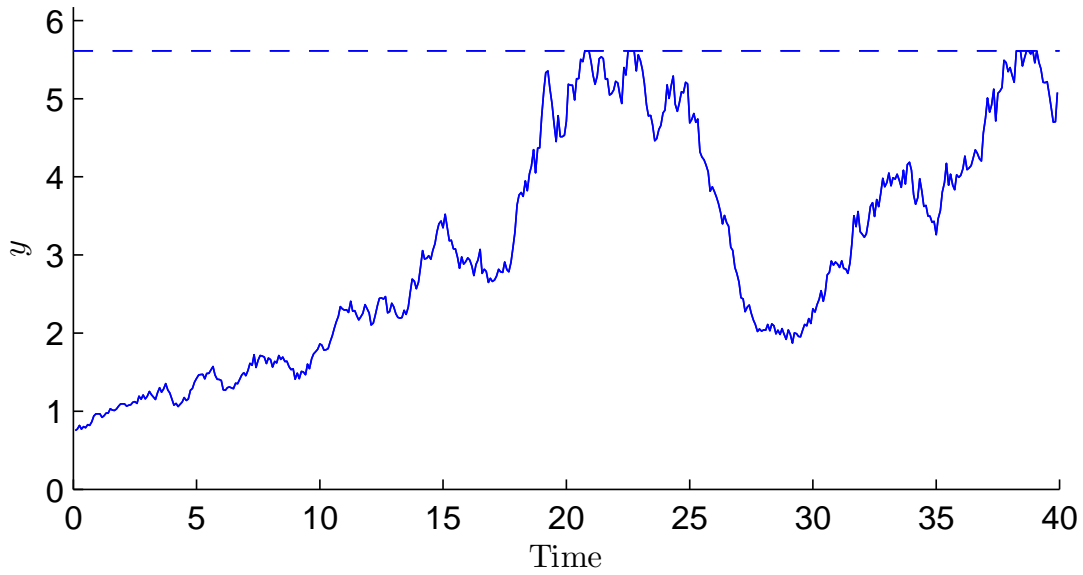
Panel B: Mass of Firms in the Economy
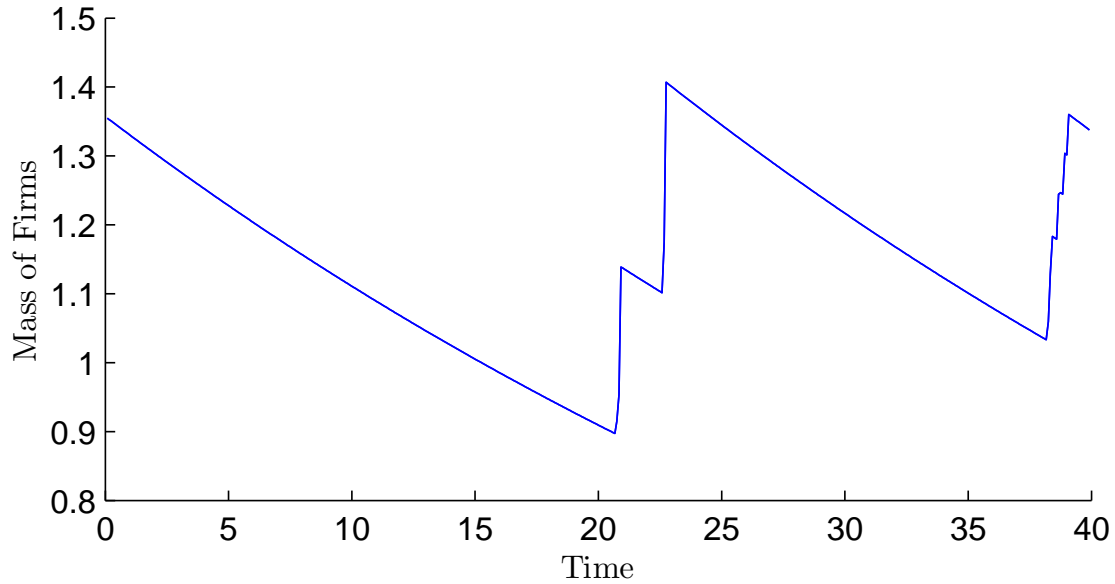
Figure 2. Sample Firms Dynamics

This figure shows, for three randomly selected firms from the simulation in Figure 1, sample paths of the x-shocks (Panel A), firm values (B), book-to-market ratios (C), price-earnings ratios (D), as well as betas with respect to y (E) and with respect to the aggregate market (F). The vertical lines indicate the times of exercise of the idiosyncratic options.

Panel A: Three Sample Paths of Idiosyncratic Shocks x_i
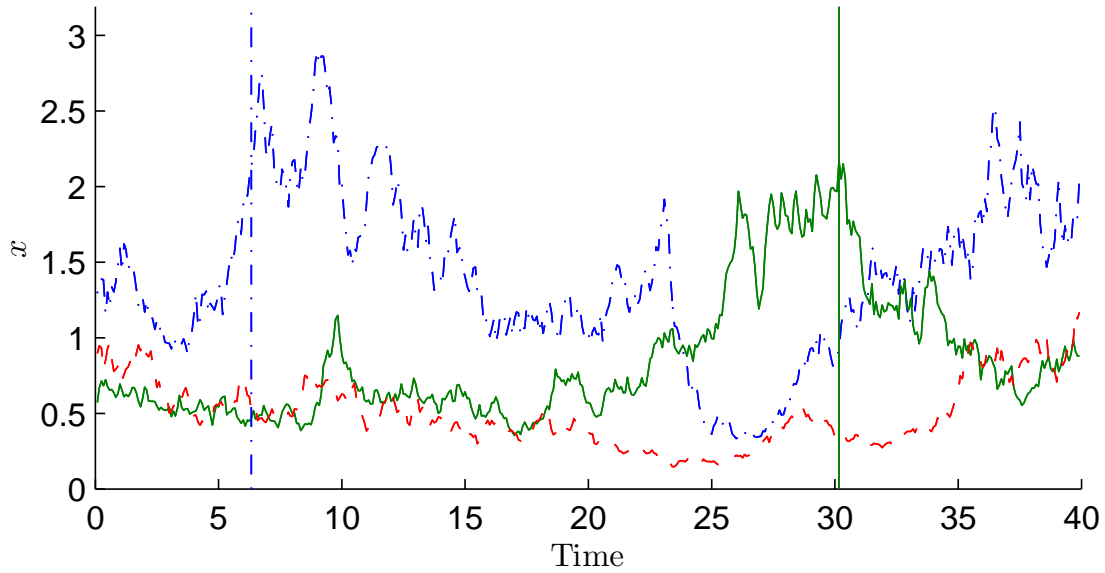
Panel B: Corresponding Firm Values
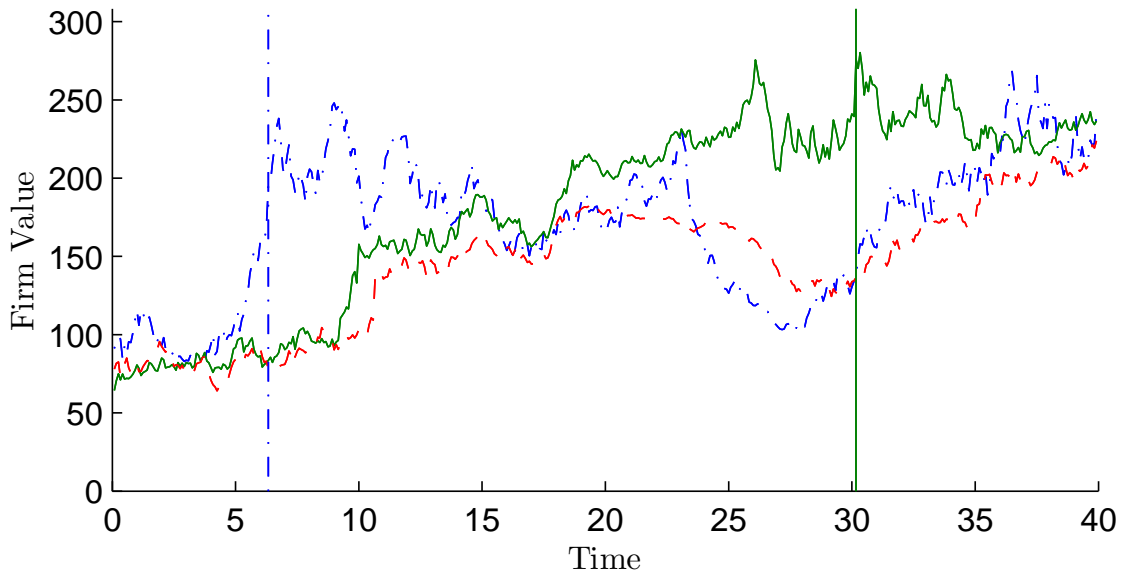
Figure 2. continued

Panel C: Corresponding Book-to-Market Ratios
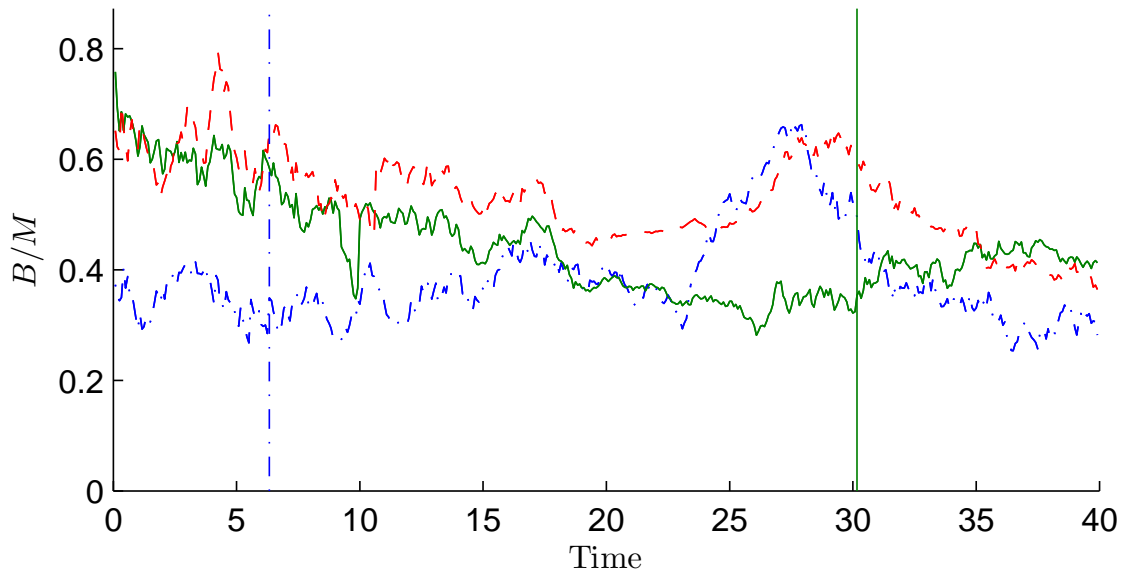
Panel D: Corresponding Price-Earnings Ratios
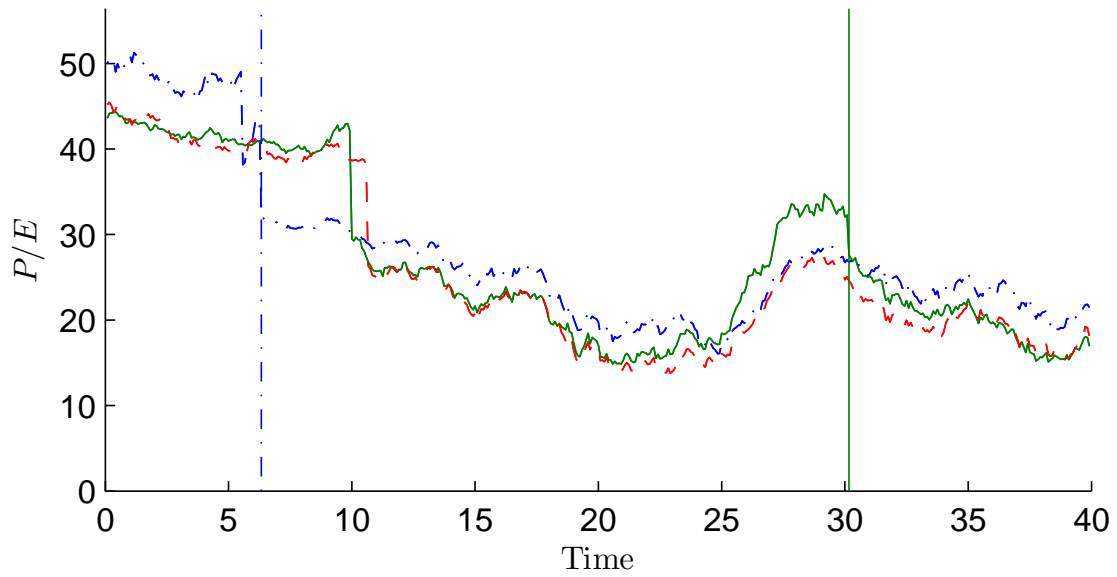
Figure 2. continued

Panel E: Corresponding Firm Factor Betas
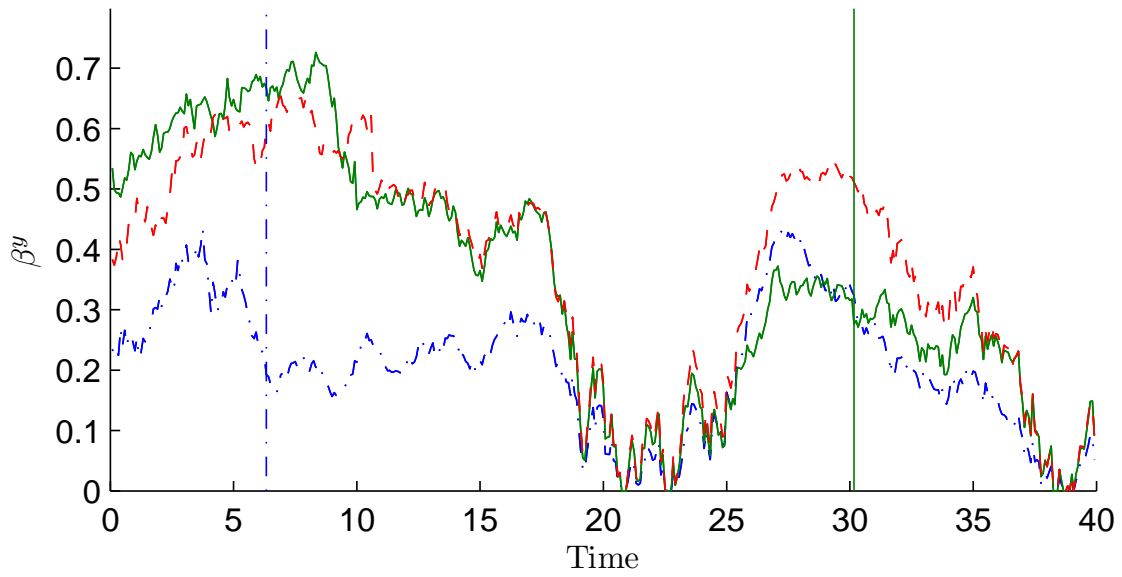
Panel F: Corresponding Firm Market Betas
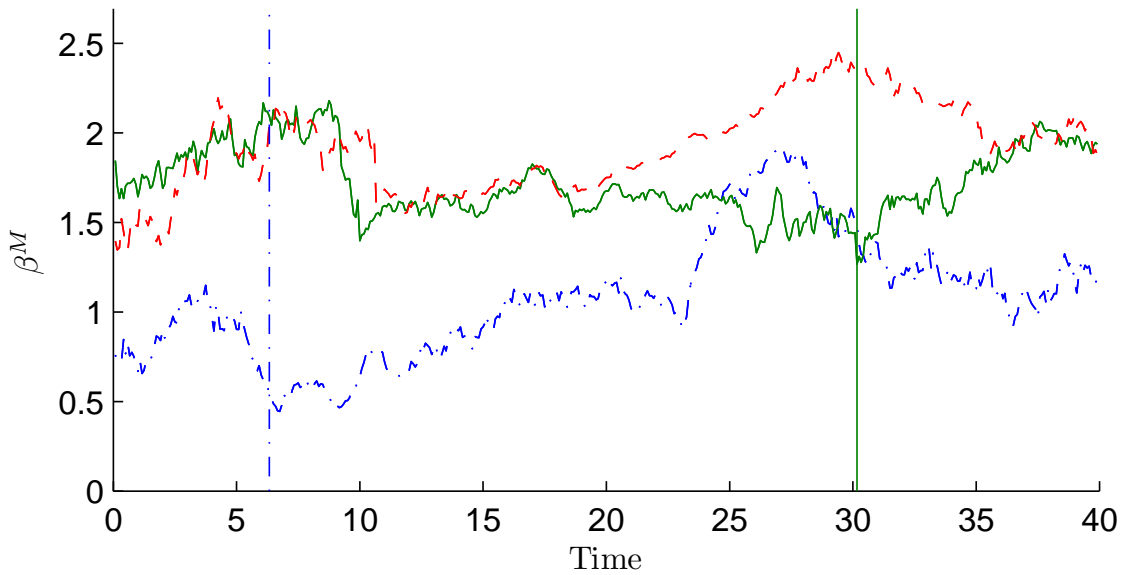
Figure 3. Book-to-market and Size Sorted Portfolio Returns

This figure plots average returns of characteristic sorted portfolios. In Panel A, 20 portfolios are formed based on book-to-market ratio, and returns are plotted against average log book-to-market characteristics. Panels B – D show the results for market capitalization, price-earnings ratio, and idiosyncratic volatility.

Panel A: Returns of Book-to-Market Portfolios
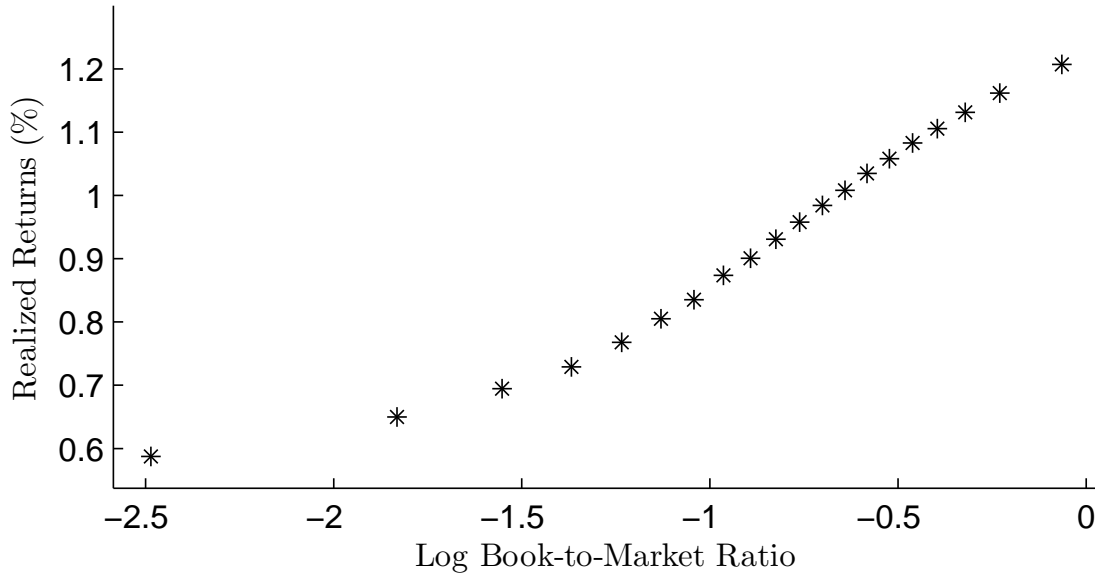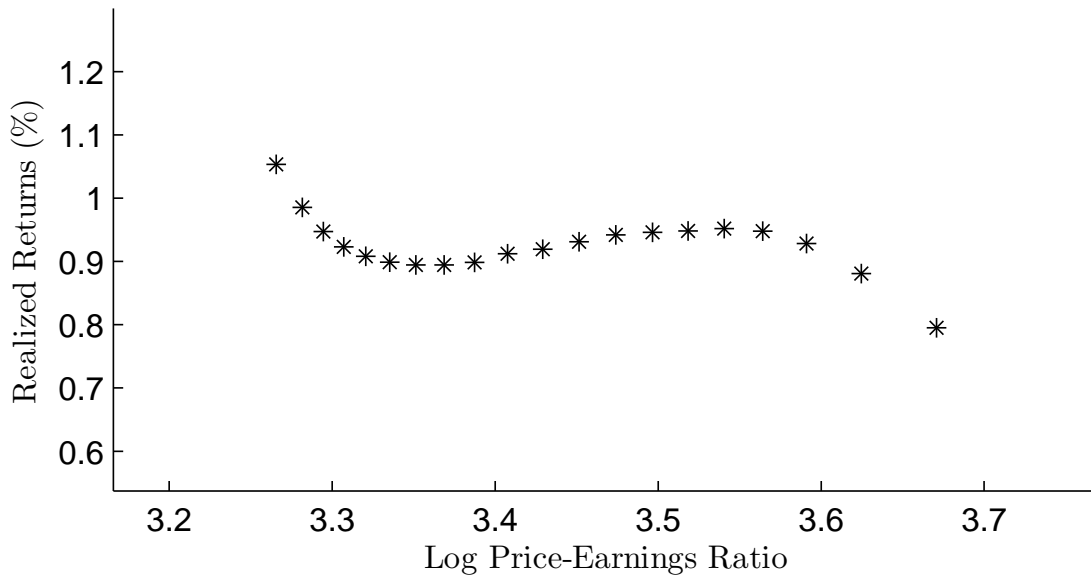

Panel B: Returns of Size Portfolios
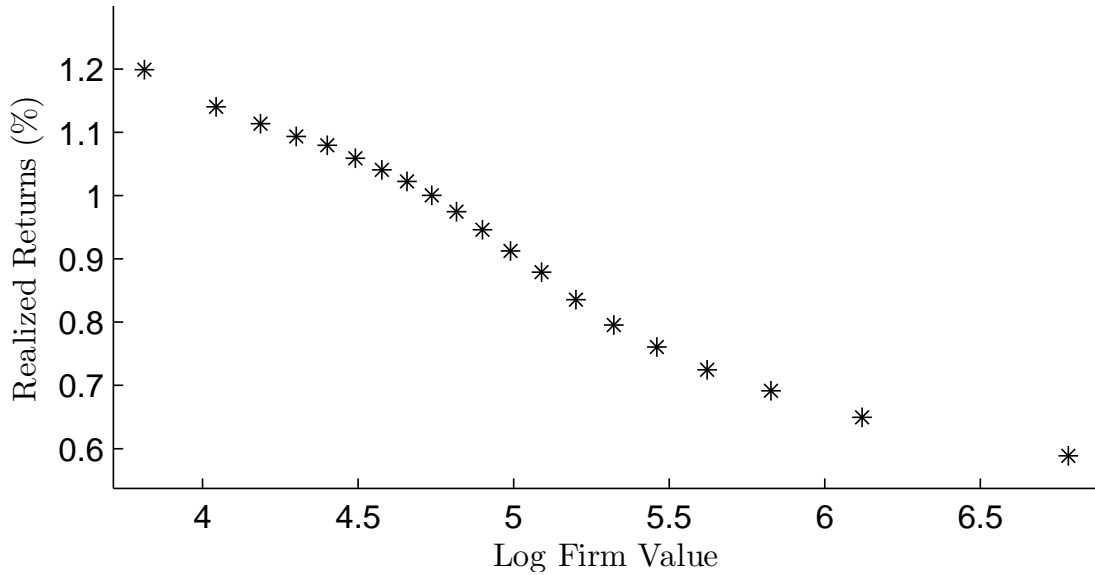
Figure 3. continued

Panel C: Returns of Prices-to-Earnings Portfolios

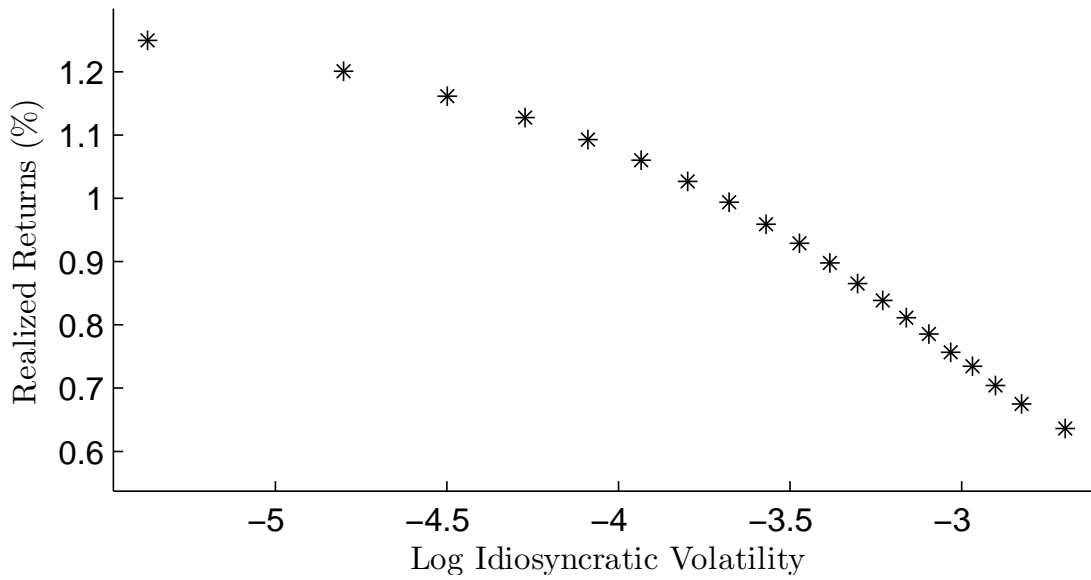Panel D: Returns of Idiosyncratic Volatility Portfolios

Figure 4. Firm Characteristics around Option Exercise

This figure plots average firm value (top row), book-to-market ratio (middle row), and betas (bottom row) in a 48-month window around exercise of the y-option (Panel A) and x-option (B).
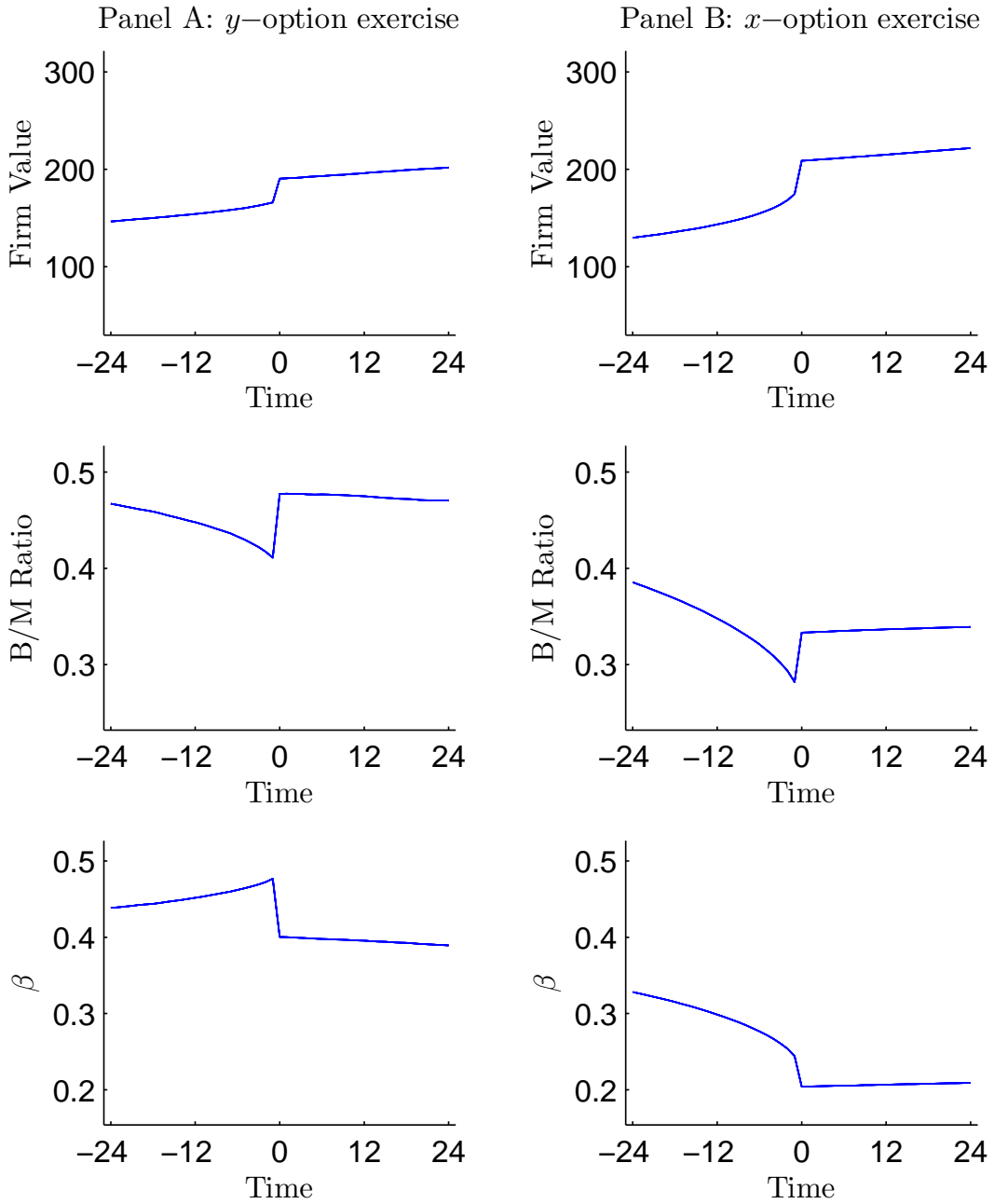
Figure 5. Time Variation in Risk and Value Premium

This figure plots the market risk premium (MRP) in percent annually, a measure of the cross-sectional variation in factor betas ($\beta_{10}^y - \beta_1^y$) and market betas ($\beta_{10}^M - \beta_1^M$), and the realized value premium (VP) against the level of the systematic shock y. The variation in betas is measured as the difference between the average betas of the first and tenth decile of beta-sorted portfolios. The realized value premium is computed as the difference in returns of the top and bottom decile of book-to-market sorted portfolios. Panel A shows the results for the case without competition and entry of new firms. Panel B is the fully specified case.