# Dynamic Bank Runs*

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**Abstract**

We develop a dynamic model of bank runs. A bank finances its long-term investment by rolling over short-term debts with a continuum of creditors, whose contract periods are asynchronous. In deciding whether to roll over the debt, each creditor faces the future rollover risk of the bank with other creditors, i.e., the bank fundamental could fall during his contract period, causing other maturing creditors to withdraw money and forcing the bank to liquidate its asset prematurely. Different from the static bank-run models with multiple equilibria, we derive a unique monotone equilibrium, in which creditors coordinate their asynchronous rollover actions based on observable fundamental shocks. Our model captures a central element of the ongoing financial crisis–even in the absence of any fundamental deterioration, fluctuations in the capital markets such as small changes in the volatility and liquidation value of the bank asset, because of their roles in determining the bank's future rollover risk, could trigger preemptive runs by creditors on a solvent bank.

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*PRELIMINARY.

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# 1 Introduction

The financial crisis of 2007-2008 highlights a new form of bank run risk faced by modern financial institutions. In recent years, commercial banks, investment banks and other financial institutions increasingly rely on rolling over short-term commercial papers and overnight repo transactions to finance their investment positions in long-term risky assets such as mortgages. This type of rollover financing makes these institutions particularly vulnerable to disruptions in the capital markets, which determine the market prices of these institutions' long-term assets and therefore their ability to roll over their short-term financing. One salient example is the failure of Bear Stearns in mid-March 2008, after Bear Stearns' short-term creditors rushed to withdraw their funding, resulting in a forced sale of Bear Stearns to J.P. Morgan Chase.[1] In fact, the inability to roll over the short-term liabilities, such as overnight repos, has been attributed to as one of the direct causes that had led to the collapse of a significant part of the investment banking system in US.[2] Interestingly, commercial banks also had similar problems, as revealed by the failure of UK bank Northern Rock, another high profile casualty of the financial crisis.[3] In spite of the television images of long lines of depositors outside its branch offices, its demise was ultimately caused by the failure to roll over a substantial fraction of its short-term financing from institutional investors.

The recent crisis especially illustrates the dynamic nature of the financial institutions' rollover risk, i.e., the risk that a borrower could not raise new funds to repay maturing short-term debts (Bernanke, 2009). Individual creditors, by anticipating possible difficulty of an institution to roll over its short-term debt in the future, may decide to run on the institution now even if its current fundamental is still healthy. This type of preemptive runs, which is the central theme of this paper, lies at the heart of the challenges confronting the great efforts by the governments and central banks to restore the world financial system. The standard bank-run models, e.g., Diamond and Dybvig (1983) and Goldstein and Pauzner (2005) are all in static settings, and therefore not suited for analyzing dynamic bank runs. In this paper, we develop a dynamic model of bank runs and apply the model to discuss several issues related to the interaction between the capital markets and the financial institutions' rollover risk.

We build a parsimonious model in continuous time. A bank, which shall be broadly

[1] For a detailed description, see the letter from the former SEC Chairman Christopher Cox to the Basel Committee, which is available at http://www.sec.gov/news/press/2008/2008-48htm.

[2] See Brunnermeier (2009) for an overview of the financial crisis of 2007-2008.

[3] See Shin (2009) for a vivid account of this episode.

interpreted as a commercial bank, an investment bank or an investment firm alike, finances its long-term investment position by issuing short-term debt to a continuum of creditors. The bank's asset fundamental fluctuates randomly over time and is observable to the public at any time. While the debt is short term, the contract lasts for a period of time, during which the creditor's money is locked in and only upon the end of which the creditor has the option to either roll over the debt or to withdraw the money. A key feature of our model is that the creditors' contract periods are asynchronous, i.e., the contract expirations are spread out across time. This realistic feature implies that when each creditor makes his rollover decision, he faces the uncertainty about other creditors' decisions during his contract period. i.e., other creditors might choose to withdraw and forcing the bank to liquidate its asset prematurely. This coordination problem among creditors who make their decisions at different times could lead to self-fulfilling multiple equilibria–in the same spirit of Diamond and Dybvig (1983)–if the bank's asset fundamental is constant and within an intermediate region. In the good equilibrium, each creditor chooses rollover after his current contract matures anticipating others to do so too in the future, while in the bad equilibrium each creditor chooses withdrawal expecting others to withdraw too in the future.

However, when the bank's asset fundamental is time-varying, we are able to derive a unique monotone equilibrium building on an important insight from the dynamic game literature (e.g., Frankel and Pauzner, 2000) that the creditors can coordinate their expectations of the future equilibrium outcome based on the current bank fundamental. In this equilibrium, each creditor chooses to roll over his debt if the bank fundamental is above a certain threshold. Despite the absence of the multiple equilibria, a preemptive bank run could occur through a rat race between the creditors in choosing their optimal rollover thresholds. Anticipating that the bank fundamental might deteriorate in the future and drop below other maturing creditors' rollover threshold, each creditor would use a high threshold to protect himself, which in turn motivates other creditors to use an even higher threshold. This rat race could lead creditors to eventually use a threshold substantially higher than the reasonable fundamental needed to justify the solvency of the bank. In other words, creditors could choose to run on a fundamentally solvent bank.

This preemptive bank run ultimately arises from the bank's fundamental risk, exacerbated by the illiquidity of the capital markets. As recognized by many pundits, both factors have contributed to the recent financial crisis. The unique monotone equilibrium derived in our model makes it possible to examine the joint effects of these two factors on the financial institutions' rollover risk. Our model shows that the capital market illiquidity can substantially exacerbate the rollover risk. Intuitively, a lower liquidation value of the bank asset exposes a creditor to a greater expected loss in the event that during the creditor's contract period the bank fails to roll over its debts with other maturing creditors and have to liquidate its asset prematurely. As a result, in making his rollover decision, the creditor would choose a higher rollover threshold to protect himself, thus making the bank more vulnerable to runs. Similarly, a higher volatility of the bank asset also exposes each creditor to a greater rollover risk of the bank because during the creditor's contract period the bank fundamental is now more likely to hit below the other creditors' rollover threshold. This effect in turn motivates each creditor to use a higher rollover threshold in making his rollover decision, again making the bank more vulnerable to runs.

Through the rollover risk channel, our model captures the evident vulnerability of modern financial institutions displayed in the recent financial crisis to fluctuations in the external capital markets–even in the absence of any fundamental deterioration, small price decreases or volatility increases in the secondary markets of assets held by the financial institutions could trigger preemptive runs by creditors on solvent institutions. Our model thus explains a puzzling phenomenon in the recent crisis–the suddenly disappearing debt capacity of many financial institutions (such as Bear Stearns and Lehman Brothers) even though many pundits had argued that their asset fundamentals right before their collapses were still healthy. On policy implications for the governments and central banks to restore the world financial system, our model points out stabilizing the asset markets, which serves the role of improving liquidation value and reducing price volatility of the assets of those distressed financial institutions, as the key solution. Our model also provides a tractable framework for quantitative evaluations of various government policies, such as the initial Troubled Asset Relief Program (TARP) proposed by the former Treasury Secretary Henry Paulson for the government to buy out troubled assets from the balance sheet of those institutions.

Given the bank's rollover risk, our model also shows that each individual creditor prefers a shorter debt maturity so that he has the option to pull out before others when the fundamental is falling. Thus, in the absence of any commitment devise like debt covenants or regulatory requirement, both the bank and the creditor can gain from reducing the maturity of the contract without affecting the overall probability of the bank failure. Since this argument applies to every creditor, it can trigger another rat race between the creditors, in addition to the one in choosing rollover threshold. This maturity rat race could substantially reduce the bank's debt maturity, and thus explains why short-term financing becomes more and more pervasive— to some extent overly used— by financial institutions.

Our model also has an important implication for credit risk modeling. The standard approach, following the classic structural models of Merton (1974) and Leland (1994), determines a firm's credit risk by the probability and loss in the event that the firm's asset value drops below the debt value or that the firm's equity drops to zero. This approach, however, ignores an important source of credit risk–the firm's rollover risk. Like banks, industrial firms often rely on a large pool of public investors to fund their debt financing and thus need to regularly raise new debts to retire maturing debts. As a result, they are also affected by the potential coordination problem between different investors in the form of the rollover risk modeled in this paper. According to our model, investors' anticipation of a firm's difficulty to roll over its debts in the future, for reasons related to either the firm's fundamental or the capital market liquidity, could already affect the firm's ability to raise capital now. More precisely, the inability of investors to perfectly coordinate their rollover decisions leads to a higher default threshold in the firm's asset value than what the standard models imply. Our model thus helps resolve the challenge faced by the standard structural models in explaining the large credit spreads observed in the data.

The paper is organized as the following. Section 2 reviews the related literature. Section 3 describes the model setup. We derive the unique monotone bank-run equilibrium in Section 4, and provide several comparative statics results in Section 5. Section 6 discusses various implications of the model. Finally, Section 7 concludes the paper and provides some further discussions on the model. All the technical proofs are given in the Appendix.

# 2 The Related Literature

In the classic bank run model of Diamond and Dybvig (1983), depositors of a bank simultaneously choose whether to withdraw their money or to stay for long-term. Because the depositors' collective withdrawal can force the bank to liquidate its long-term asset prematurely and depositors are not able to coordinate their actions, two self-fulfilling equilibria emerge. In the good equilibrium, all the depositors choose to stay for the long-term, while in the bad equilibrium, they all demand early withdrawal. In the recent literature, Goldstein and Pauzner (2005), by adopting the global games framework developed in the game theory literature (e.g., Morris and Shin, 2003), derive a unique bank-run equilibrium. In this equilibrium, investors coordinate their expectations of the two possible outcomes through their private information about the unobservable bank fundamental and a panic-based bank run could still occur in certain states.

Our dynamic model shares an important feature of the global game framework–the existence of upper and lower dominance regions. In the upper (lower) dominance region, the bank fundamental is sufficiently high (low) so that each creditor's dominant strategy is rollover (withdrawal) regardless of other creditors' strategy. Thus, the creditors' equilibrium strategy is uniquely determined in the two dominance regions. Our model also builds on an important insight from the dynamic game studied by Frankel and Pauzner (2000) that time-varying fundamental can help agents coordinate their expectations of future equilibrium outcome. More precisely, our model uses two important ingredients to ensure a unique monotone equilibrium. First, the creditors' asynchronous debt periods spread their rollover decisions uniformly across time, and thus avoiding the coordination problem between creditors who make their decisions at the same time. Second, the publicly observable bank fundamental is time-varying and for sure to hit at least one of the dominance regions. As a result, the creditors can backwardly induce the equilibrium in the intermediate region between the two dominance regions based on the unique equilibrium outcomes at the two ends of the region as boundary conditions.

Due to the realistic debt payoffs in our bank run setting, the endogenous payoff for bank creditors does not guarantee the property of global strategic complementarity commonly assumed in the global game literature and by Frankel and Pauzner (2000). However, we are able to explicitly construct a monotone equilibrium in closed form and then verify that this equilibrium is unique. More importantly, while Frankel and Pauzner (2000) treat the fundamental shocks as a technical tool for ensuring a unique equilibrium, our model shows that they are also a key economic factor in determining the bank's rollover risk.

Guimaraes (2006) applies the Frankel-Pauzner framework to derive a unique dynamic equilibrium of currency attacks. He shows that small frictions on speculators' ability to change their positions instantaneously can cause large delay in their attack of an overvalued currency. In his model, the source of the frictions is ambiguous. In contrast, in our bank-run model, the frictions emerge naturally from the lock-in effect of the creditors' debt contracts. Thus, our model can directly relate financial institutions' rollover risk to their debt structure, and, furthermore, shows that each creditor's anticipation of the future rollover risk can lead to the use of shorter and shorter debt maturity.

A recent paper by Acharya, Gale, and Yorulmzer (2009) also considers the rollover risk faced by financial institutions, and shows that under certain information structure the debt capacity of a given long-term asset can shrink to zero as rollover frequency increases to infinity. In contrast, we emphasize the asynchronous nature of the short-term funding and the intrinsic coordination problem among creditors, and information structure does not play any role.

*** Literature Review Incomplete ***

# 3 Model

We consider a continuous-time model with infinite time horizon. A bank invests in a long-term asset by rolling over short-term debts with a continuum of creditors. Each of the creditors is risk neutral and discounts future cashflows at a rate $\rho > 0$.

## 3.1 Asset

We normalize the unit of the bank's asset holding to be 1. The asset generates a cashflow of $rdt$ in the time interval $[t, t + dt]$. At a random time $\tau_\phi$, which arrives according to a Poisson process with parameter $\phi > 0$, the asset matures and provides a final payoff. An important advantage of assuming a random asset maturity with a Poisson process is that at any point before the maturity, the expected remaining maturity is always $1/\phi$.

The asset's final payoff is equal to the time-$\tau_\phi$ value of a stochastic process $y_t$, which follows a geometric Brownian motion:

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ$$

with constant drift $\mu$ and volatility $\sigma > 0$. We assume that the value of the fundamental process is publicly observable at any time. Before the asset matures at $\tau_\phi$, the bank could also liquidate it at a price

$$L(y_t) = L + l y_t,$$

where $L \in (0, 1)$ and $l > 0$ is small. We also assume that the liquidation decision is irreversible and we do not allow for partial liquidation.

We broadly interpret the bank asset either as a long-term real investment position or as a long-term illiquid financial asset. The process $y_t$ represents the public agents' expectation of the bank asset's long-term fundamental value, while $L(y_t)$ represents the scrap value of liquidating a real asset or as the fire sale price of liquidating an illiquid financial asset on the secondary market. In the latter case, the price discount depends on the secondary market liquidity.

## 3.2 Debt Financing

We emphasize two important features of a real-life bank's debt structure. First, there is a duration mismatch between asset and liability, i.e., a bank usually finances its long-term investment position by rolling over short-term debts. Second, a bank usually has many creditors whose contract periods are asynchronous. More specifically, each creditor is locked in by his debt contract to fund the bank for at least a period of time. For example, the overnight repo transactions span one day, while commercial paper goes from one to 270 days. Only at the end of the contract period, the creditor can decide whether to roll over the debt or to withdraw money. The contracts of different creditors expire at different times. This is a realistic feature because different divisions of a bank usually finance their separate operations using different instruments and through different counterparties in a decentralized manner. As a result, the financing contracts used by different divisions naturally expire at different times.

To capture these realistic features, we assume that in our model, the bank finances its asset holding by issuing one unit of short-term debt equally among a continuum of creditors with measure 1. The promised interest rate is $r$ so that the cashflow from the asset exactly pays off the interest payment until the asset matures or when the bank is forced to liquidate the asset prematurely. Once a creditor lends money to the bank, the debt contract lasts for a random period, which ends upon the arrival of an independent Poisson shock with parameter $\delta > 0$. In other words, the duration of each debt agreement has an exponential distribution and the distribution is independent across different creditors. Once the contract expires, the creditor chooses whether to roll over the debt.

While the random duration assumption appears different from the standard contract with a predetermined maturity, it greatly simplifies the complication involved in dealing with the debt's maturity effect and at the same time maintains the aforementioned key features of a real-life bank's debt structure. With a random debt duration with exponential distribution, at any time before the debt maturity, the expected remaining maturity is always $1/\delta$. When the value of $1/\delta$ is matched with the fixed maturity of a real-life debt contract, this assumption captures the first order effect of debt maturity on a creditor's rollover decision at the contract rollover point,[4] although it may not be effective for valuing the debt contract when it is already partially inside the contract period. The random maturity assumption of each individual debt contract does not affect the aggregate debt structure of the bank–in aggregation the bank always has a fixed fraction of its debt contracts maturing over time, which exactly captures the asynchronous structure of a real-life bank's outstanding debts.

Over a short time interval $[t, t + dt]$, $\delta dt$ fraction of the bank's debt contracts expire and these creditors have the option to either roll over their debts or to withdraw their money. For simplicity, we do not explicitly model the bank's liquidity reserve or credit lines with other banks. Instead, we assume that if some maturing creditors choose not to roll over their debts, the bank needs to find new creditors to replace them and subsequently there is a probability that the bank may not find a sufficient number of new creditors and would be forced to liquidate its asset prematurely and file for bankruptcy. This assumption implies that the capital markets are illiquid–the bank cannot find a single large creditor to finance all of its debts and thus to avoid the coordination problem among heterogeneous creditors; neither could it always find a sufficient number of new creditors to replace outgoing creditors even when the bank fundamental is healthy.

More specifically, we assume that the probability of such bankruptcy increases with the flow of outgoing creditors. That is, if all of the maturing creditors during the time interval choose to withdraw their money, the probability of bankruptcy is $\theta\delta dt$, where $\theta > 0$ is the parameter that measures the instability of the bank. The higher the value of $\theta$, the more likely the bank will be liquidated by the same rate of creditor outflow. One can view this assumption as a reduced form to capture either the bank's technology to search for new creditors, or as the financial strength of the bank's balance sheet (i.e. how much credit line the bank has from other banks and how much liquidity reserve it has).[5] A similar modeling choice is also used by Morris and Shin (2004) to study firms' credit risk.

Once the bank liquidates its asset, the liquidation value will be used to pay off the remaining creditors on an equal basis. In other words, all the creditors who are locked in by

[4] This assumption also generates an artificial second-order effect–it is possible for a creditor to be released early and therefore to withdraw before other creditors when the asset fundamental deteriorates. This option makes the creditor less worried about the bank's rollover risk than he would be if his debt contract has a fixed maturity. This in turn makes him more likely to roll over his debt. Thus, by assuming the random debt maturity, our model underestimates the bank's rollover risk.

[5] There are several reasons to justify why some new creditors are willing to take over the debts of outgoing creditors. In reality, creditors often need to exit their profitable positions for liquidity reasons, instead of deteriorating bank fundamentals. The existence of liquidity driven creditor outflow provides a rational for uninformed new creditors to come in. A bank's credit lines with other institutions provide another source of new creditors.

their current contracts will get the same payoff determined by the liquidation value of the bank's asset. Only those who had chosen to withdraw earlier get a full pay.

While we treat the creditors' rollover frequency $\delta$ (or equivalently the expected debt maturity) as given for most of our analysis, we will analyze the creditors' preference over debt maturity and endogenize rollover frequency in Section 5.3. To focus on the coordination problem between creditors, we also take the interest payment of the debts as given and leave a more elaborate analysis of the effects of endogenous interest payments for future research.

### 3.3 Parameter Restrictions

To make the analysis meaningful, we impose several parameter restrictions. First, we bound the interest payment by

$$\rho < r < \rho + \phi. \tag{1}$$

Second, we limit the growth rate of the asset fundamentals by

$$\mu < \rho + \phi + (1 + \theta)\, \delta, \tag{2}$$

because, otherwise, the asset value would explode. Third, we also limit the liquidation value of the bank asset:

$$L + l < 1, \tag{3}$$

so that the premature liquidation of the asset leads to a price discount. Finally, we assume that the debt contract matures reasonably fast:

$$\delta > \frac{\phi}{\theta\,(1 - L - l)}. \tag{4}$$

## 4 The Bank-Run Equilibrium

Given the bank's financing structure described in the previous section, we now analyze the bank-run equilibrium. We limit our attention to monotone equilibria, that is, symmetric equilibria in which each creditor's rollover strategy is monotonic with respect to the bank fundamental $y_t$ (i.e., to roll over the debt if the bank fundamental is above a threshold). In making the rollover decision, a creditor rationally anticipates that once he rolls over the debt, then during the following contract period, volatility could cause the bank fundamental to fall below other creditors' rollover threshold and thus exposing him to the bank's rollover risk. As a result, the creditor's optimal rollover threshold depends on other creditors' threshold choices and the bank's fundamental volatility.

In this section, we first set up an individual creditor's optimization problem in choosing his optimal threshold. Then, we discuss a central feature of the bank-run equilibrium– creditors could engage in a rat race in choosing the rollover threshold, i.e., each creditor sets a high threshold to protect himself against the bank's rollover risk with other maturing creditors in his lockup period, which in turn motivates other creditors to choose an even higher threshold. We then construct a unique symmetric equilibrium in closed form. In this equilibrium, every creditor chooses the same threshold. We also characterize the key ingredients that lead to the unique equilibrium. To highlight the rollover risk caused by the coordination problem between the creditors, we also provide a benchmark model with a single creditor who provides all the financing to the bank.

## 4.1 An Individual Creditor's Problem

We first analyze an individual creditor's problem. We assume that the creditor holds a small fraction of the bank's outstanding debts and that all other creditors use a monotone strategy with a rollover threshold $y_*$ (i.e., they will roll over their contracts if and only if the bank fundamental is above $y_*$ at the time when their contracts expire). During the creditor's contract period, his value function depends directly on the bank fundamental $y_t$, and indirectly on other creditors' rollover threshold $y_*$. We denote $V(y_t; y_*)$ as the creditor's value function normalized by the unit of debt he holds.

For each unit of debt, the creditor receives a stream of interest payment $r$ until a random time $\tau$,

$$\tau = \min(\tau_\phi, \tau_\delta, \tau_\theta)$$

which is the earliest of the following three events: the asset matures at a random time $\tau_\phi$, the creditor's own contract expires at $\tau_\delta$, or some of the other creditors choose not to roll over their contracts and thus forcing the bank to liquidate the asset prematurely at $\tau_\theta$. Figure 1 illustrates these three possible outcomes to the creditor at the end of three different fundamental paths. On the top path, the bank stays alive until its asset matures at $\tau_\phi$. At this time, the creditor gets a final payoff of $\min\left(1, y_{\tau_\phi}\right)$, i.e., the face value 1 if the asset's maturity payoff $y_{\tau_\phi}$ is sufficient to pay off all the debts, and $y_{\tau_\phi}$ otherwise. On the bottom path, the bank fundamental drops below the creditors' threshold and the bank is eventually forced to liquidate its asset at $\tau_\theta$ before the creditor's contract expires. At this time, the creditor gets $\min(1, L + ly_{\tau_\theta})$, i.e., the face value 1 if the asset's liquidation value $L + ly_{\tau_\theta}$ is sufficient to pay off all the debts, and $L + ly_{\tau_\theta}$ otherwise. On the middle path, while the

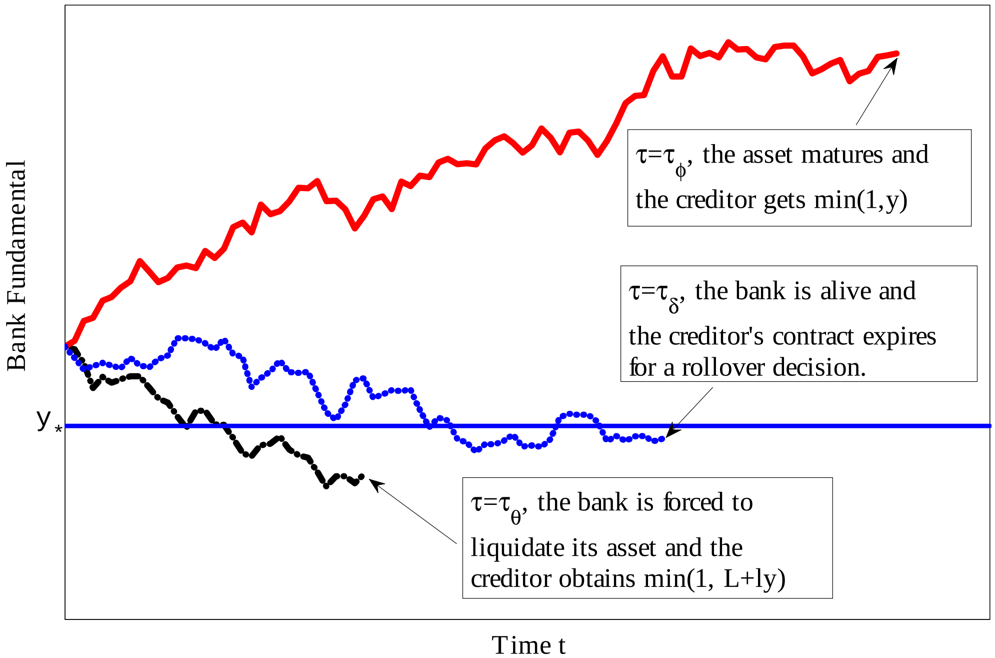


Figure 1: Three possible outcomes to a creditor

bank stays alive (although its fundamental dips below other creditors' rollover threshold on the path) before $\tau_\delta$ when the creditor's contract expires. At this time, the creditor chooses whether to roll over the debt depending on whether the continuation value $V(y_{\tau_\delta}; y_*)$ is higher than getting the one dollar back.

Thus, the creditor's value function is given by

$$\begin{aligned} V(y_t; y_*) &= E_t\{\int_t^\tau e^{-\rho(s-t)} r ds + e^{-\rho(\tau-t)}[\min(1, y_\tau)\,\mathbf{1}_{\{\tau=\tau_\phi\}} \\ &\quad + \min(1, L + l y_\tau)\,\mathbf{1}_{\{\tau=\tau_\theta\}} + \max_{\text{rollover or run}}\{1, V(y_\tau; y_*)\}\,\mathbf{1}_{\{\tau=\tau_\delta\}}]\} \end{aligned}$$

where $\mathbf{1}_{\{\cdot\}}$ is an indicator function, which takes a value of 1 if the statement in the bracket is true and zero otherwise. The individual creditor's future payoff during his contract period depends on other creditors' choices because of the possibility that other creditors' withdrawals can force the bank to liquidate its asset prematurely. This dependence leads to complimentarity in the creditors' rollover decisions, and therefore a coordination problem between creditors who make their rollover decisions at different times.

Also note that when the bank fundamental $y_t$ is sufficiently low (i.e., close to zero), an individual creditor's dominant choice is withdrawal. This is because that even if all other creditors choose to roll over in the future, the expected asset payoff at the maturity plus the interest payments before the asset maturity are not as attractive as getting one dollar back now. On the other hand, when the bank fundamental $y_t$ is sufficiently high (i.e., sufficiently higher than $(1-L)/l$), an individual creditor's dominant choice is rollover. This is because that even if all other creditors choose to withdraw in the future, the asset's liquidation value is sufficient to pay off the debts in the event of a forced liquidation. These two regions are often called the lower and upper dominance regions. Their existence is important for ensuring a unique equilibrium.

By considering the change of the creditor's value function over a small time interval $[t, t+dt]$, we can derive his Bellman equation:

$$\begin{aligned} \rho V\left(y_t; y_*\right) &= \mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy} + r + \phi\left[\min\left(1, y_t\right) - V\left(y_t; y_*\right)\right] \\ &\quad + \theta\delta \mathbf{1}_{\{y_t < y_*\}}\left[\min\left(L + l y_t, 1\right) - V\left(y_t; y_*\right)\right] + \delta \max_{\text{rollover or run}} \left\{1 - V\left(y_t; y_*\right), 0\right\}. \end{aligned} \tag{5}$$

The left-hand side term $\rho V\left(y_t; y_*\right)$ represents the creditor's discounting of his future value function using the discount rate $\rho$. This term should be equal to the expected increase in his value function, as summarized by the terms on the right-hand side. The first two terms on the right-hand side $\mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy}$ capture the expected change in the value function caused by the fluctuation in the bank fundamental $y_t$. The third term $r$ is the interest payment per unit of time. The forth term $\phi\left[\min\left(1, y_t\right) - V\left(y_t; y_*\right)\right]$ captures the possibility that the asset matures during the time interval, which occurs at a probability of $\phi dt$ with an impact of $\min\left(1, y_t\right) - V\left(y_t; y_*\right)$ on the creditor's value function. The fifth term represents the expected effect when the bank is forced to liquidate the asset by other creditors' runs, which occurs at a probability of $\theta\delta \mathbf{1}_{\{y_t < y_*\}} dt$ (the other creditors will only run if $y_t < y_*$) with an impact of $\min\left(L + l y_t, 1\right) - V\left(y_t; y_*\right)$ on the creditor's value function. The last term captures the expected effect from the expiration of the creditor's own contract, which arrives at a probability of $\delta dt$. Upon its arrival, the creditor chooses whether to rollover or to run: $\max_{\text{rollover or run}} \left\{1 - V\left(y_t; y_*\right), 0\right\}$.

Note that from any individual creditor's view, the probability of the event that his contract matures and the bank is forced into a premature liquidation is in the second order of $(dt)^2$. Thus, whether the creditor gets 1 or the asset liquidation value in such an event is inconsequential. In addition, once the forced liquidation occurs, all the remaining creditors have the same priority in dividing the bank's liquidation value.[6]

[6] More precisely, after paying off the outgoing creditors, the remaining cashflow $L + l y_{\tau_\theta} -$

We first make an intuitive assertion that the asset's liquidation value at the equilibrium threshold $y_*$ must be less than one:

$$L + l y_* < 1.$$

Otherwise, creditors do not need to worry about others' withdrawal because the liquidation value is more than enough to cover all the debts. We will later show that this assertion holds in equilibrium. Thus, we can rewrite the Bellman equation to be

$$\begin{aligned} 0 \; = \; & \frac{\sigma^2}{2} y_t^2 V_{yy} + \mu y_t V_y - \left[\rho + \phi + \theta\delta \mathbf{1}_{\{y_t < y_*\}}\right] V\left(y_t; y_*\right) \\ & + \phi \min\left(1, y_t\right) + \theta\delta \mathbf{1}_{\{y_t < y_*\}} \left(L + l y_t\right) + r + \delta \max_{\text{rollover or run}} \left\{1 - V\left(y_t; y_*\right), 0\right\}. \end{aligned} \tag{6}$$

It is obvious that an individual creditor will only choose to roll over his contract if $V\left(y_t; y_*\right) > 1$, and withdraw money otherwise. This implies that the creditor's optimal threshold $y'$ satisfies $V\left(y'; y_*\right) = 1$. In a symmetric equilibrium, we must have $y' = y_*$ so that

$$V\left(y_*; y_*\right) = 1.$$

This is the condition to determine the equilibrium threshold.

## 4.2 Rat Race in Threshold Setting

The Bellman equation in (6) shows that an individual creditor's optimal threshold choice $y'$ depends on the other creditors' threshold choice $y_*$. Intuitively, if the other creditors use a higher threshold, it is more likely that the bank fundamental would hit below that threshold during the individual creditor's contract period and thus exposing the bank to greater rollover risk. Consequently, the creditor would prefer a higher threshold to protect himself. This dependence leads to a central feature of our model–creditors engage in a rat race of choosing a higher threshold, i.e., to protect himself against the bank's rollover risk with other creditors, each creditor chooses a high rollover threshold, which in turn motivates other creditors to choose an even higher threshold. When this thought process converges in the equilibrium, each creditor could end up with a threshold much higher than the necessary fundamental level to justify the solvency of the bank.

Before we formally derive the equilibrium threshold level, we illustrate this rat race in threshold setting using a simple thought experiment. Suppose that initially the bank's

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$\delta dt$ is divided among the remaining creditors $1 - \delta dt$. Thus, each remaining creditor receives $\min\left\{\left(L + l y_{\tau_\theta} - \delta dt\right) / \left(1 - \delta dt\right), 1\right\}$, which is $\min\left\{L + l y_{\tau_\theta}, 1\right\}$ after ignoring the higher order $dt$ terms.

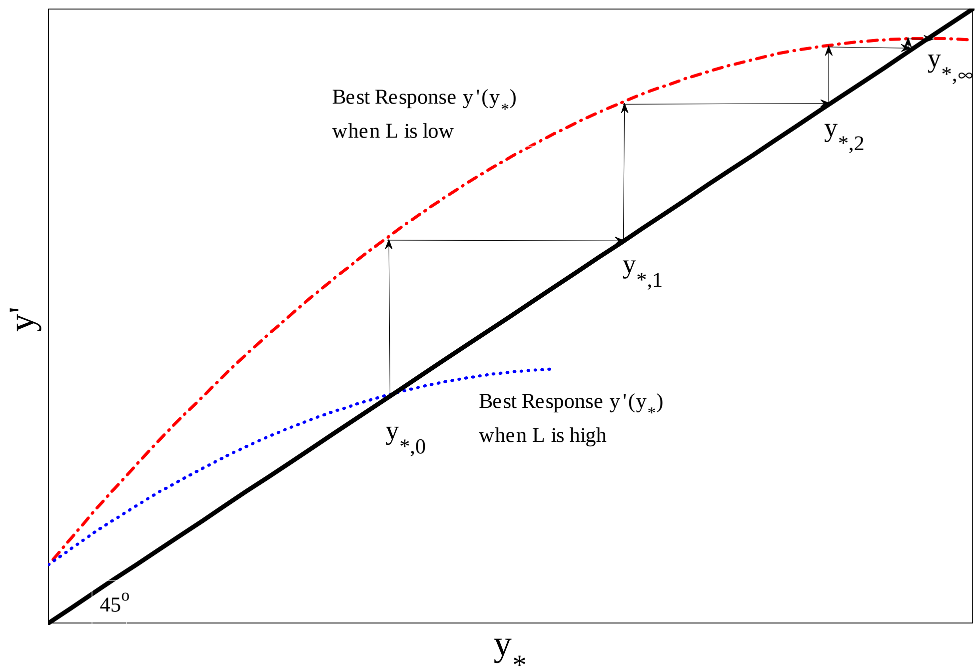


Figure 2: Rat race in threshold setting

asset liquidation value is $L_h$, and, correspondingly, every creditor uses a threshold level $y_{*,0}$. Unexpectedly, at some point in time, all the creditors find out that the asset liquidation value permanently drops to a lower level $L_l < L_h$. What would be the new equilibrium threshold? Let's start with an individual creditor's threshold choice. Suppose that all the other creditors still use the original threshold $y_{*,0}$. Then, by solving the Bellman equation in (6), we can derive the creditor's optimal threshold $y_{*,1}$, which is higher than $y_{*,0}$ because the lower liquidation value generates a greater loss to the creditor if the bank fundamental falls below $y_{*,0}$ during his contract period. The difference $y_{*,1} - y_{*,0}$ represents the additional safety margin an individual creditor would demand in the bank fundamental to offset the increased risk, under the condition that other creditors' threshold does not change. Of course, every creditor will choose a new threshold and go through the same calculation as above. If all creditors choose a threshold $y_{*,1}$, then an individual creditor's optimal threshold would be $y_{*,2}$, another even higher level than $y_{*,1}$. If all creditors choose $y_{*,2}$, then each individual creditor would go through another round of threshold updating, and so on and so forth. Figure 2 illustrates this threshold setting process and that it would eventually converge to a fixed point $y_{*,\infty}$, the new equilibrium threshold.

Note that the difference between the threshold levels $y_{*,1}$ and $y_{*,0}$ represents the necessary safety margin a creditor would demand in response to the reduced asset liquidation value if other creditors' rollover strategies stay the same. This increase in threshold is eventually magnified to a much larger increase $y_{*,\infty} - y_{*,0}$ through the rat race among creditors.

## 4.3 The Unique Monotone Equilibrium

Instead of solving the equilibrium through the convergence of the rat race described above, we directly construct a symmetric monotone equilibrium by solving the unique fixed point of the creditors' rollover threshold. More specifically, we follow a three-step approach. First, we derive an individual creditor's value function $V(y_t; y_*)$ by assuming that the creditor, along with other creditors, all use the same monotone strategy with a rollover threshold of $y_*$ from the Bellman equation in (6). Second, based on the derived value function, we show that there exists a unique fixed point $y_*$ such that $V(y_*; y_*) = 1$. Finally, we prove the optimality of the threshold $y_*$ for any individual creditor, i.e., $V(y; y_*)$ only crosses 1 once, $V(y; y_*) > 1$ for $y > 1$, and $V(y; y_*) < 1$ for $y < 1$.

We summarize the main results in the following theorem.

**Theorem 1** *There exists a unique symmetric monotone equilibrium, in which every creditor decides to roll over his debt if $y_t$ is above the threshold $y_*$ and withdraws money otherwise. The creditor's value function $V(y_t; y_*)$ is given by the following two cases: if $y_* < 1$,*

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + B_1 y_t^{\eta_1} & \text{when } 0 < y_t < y_* \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t + A_2 y_t^{-\gamma_2} + B_2 y_t^{\eta_2} & \text{when } y_* < y_t < 1 \\ \frac{r+\phi}{\rho+\phi} + A_3 y_t^{-\gamma_2} & \text{when } 1 < y_y \end{cases}$$

*and if $y_* \geq 1$,*

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + D_1 y_t^{\eta_1} & \text{when } y_t < 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + C_2 y_t^{-\gamma_1} + D_2 y_t^{\eta_1} & \text{when } 1 < y_t < y_* \\ \frac{r+\phi}{\rho+\phi} + C_3 y_t^{-\gamma_2} & \text{when } y_* < y_t \end{cases},$$

*where $\eta_1$, $\gamma_1$, $\eta_2$, $\gamma_2$, $A_2$, $A_3$, $B_1$, $B_2$, $C_2$, $C_3$, $D_1$ and $D_2$ are expressions of the model parameters and $y_*$, given in the Appendix A.1. The equilibrium threshold $y_*$ is uniquely determined by the condition that $V(y_*, y_*) = 1$.*

Theorem 1 presents a unique dynamic monotone equilibrium–when each creditor's current contract matures, he will choose not to roll over the debt if the bank fundamental is below the threshold $y_*$, and consequently exposing the bank to the possibility of a forced liquidation.

Note that an individual creditor's value function is different depending on whether the equilibrium threshold $y_*$ is higher or lower than 1, the bank's debt value. This difference is related to the payoff cap of the debt contract at 1. When the other creditors use a rollover threshold above 1, an individual creditor has little incentive to roll over his maturing debt to wait for future improvement of the bank's asset fundamental because his payoff from the asset maturity is capped at 1. This realistic payoff cap in debts makes the equilibrium rollover threshold, once it is above 1, highly sensitive to changes in the model parameters such as the liquidation value and volatility of the bank asset.

## 4.4 Understanding the Uniqueness of the Equilibrium

In the classic bank run model of Diamond and Dybvig (1983), there exist two equilibria. While our model features a similar strategic complementarity among creditors as in their model, we are able to derive a unique monotone equilibrium in Theorem 1. What is the driving force of the unique equilibrium? In this section, we discuss the contribution of two important departures of our model from the standard models: asynchronous contract periods and time-varying fundamental.

### 4.4.1 Asynchronous Contract Periods

The asynchronous contract periods are an important factor for ensuring the unique monotone equilibrium. To see this, it is useful to examine a synchronous-contract case in which all the creditors' contract periods are synchronous. More specifically, suppose that their contracts all expire at a given time, say time 0, and each creditor needs to decide whether to withdraw or to roll over onto a perpetual debt contract until the bank asset matures at $\tau_\phi$. Once a creditor chooses to roll over, he is locked in afterward. Thus, the bank does not face any rollover risk after time 0. However, at time 0, all the creditors simultaneously choose their rollover decisions, and there is a coordination problem among them similar to that in Diamond and Dybvig (1983). We formally characterize this coordination problem below.

**Proposition 2** *There exist $y_h > y_l > 0$ such that when $y_0 > y_h$, it is optimal for an individual creditor to roll over even if all the other creditors choose to withdraw; when $y_0 < y_l$, it is optimal for the creditor to withdraw even if all the other creditors choose to roll over;*

*and when* $y_0 \in [y_{l,}\ y_h]$ *, the creditor's optimal choice depends on the others', i.e., it is optimal to withdraw if the others choose to withdraw and it is optimal to roll over if the others choose to roll over.*

Proposition 2 shows that there are an upper region and a lower dominance region for the bank fundamental $y_0$. In the upper dominance region ($y_0 > y_h$), the bank fundamental is so strong that it is sufficient to pay off all the debts even in the event of a forced liquidation. Thus, an individual creditor finds rollover a dominant strategy. In the lower dominance region ($y_0 < y_l$), the bank fundamental is so poor that the expected maturity payoff of the bank asset is insufficient to cover the debts even in the absence of a forced liquidation. Thus, an individual creditor finds withdrawal a dominant strategy.

However, when the bank fundamental is between these two dominance regions, i.e., when the bank fundamental is good enough to pay off the debts if the bank asset is kept to the maturity but insufficient to cover the loss from a forced liquidation, an individual creditor's optimal rollover choice depends on the other creditors'. Put it differently, when the bank fundamental is not strong enough to sustain the withdrawals of all the other creditors, an individual creditor has to go along with the other creditors. Like in Diamond and Dybvig (1983), there are two equilibria, in one of which all the creditors choose to roll over and in the other all choose to withdraw.

Therefore, when the contract periods are synchronized, multiple equilibria could emerge. This is because the collective choice by the creditors who make their rollover decisions at the same time can swing the survival of the bank. With the asynchronous debt structure of our main model, the expiration of the bank's debt contracts is uniformly spread out over time. As we argued before, the asynchronous debt structure is realistic because a real-life financial institution typically has many debt contracts outstanding, which are issued by different divisions of the institution and expire at different times. As a result, over a small interval of time (say a day), the fraction of maturing contracts is small, which makes the collective choice of these creditors insignificant to the bank. This feature thus avoids the coordination problem among the creditors whose contract expire at the same time, but leads to a different coordination problem between the creditors whose contracts expire at different times.

### 4.4.2 Time-Varying Fundamental

We now discuss the coordination problem between creditors whose contracts expire at different times. Since each creditor is locked in by the contract for a period of time, his payoff depends on the rollover decisions of other maturing creditors during his contract period. Interestingly, this dependence could again lead to multiple equilibria if the bank fundamental is constant over time. In this subsection, we first derive these multiple equilibria in Proposition 3 and then discuss how time-varying bank fundamental prevents this type of multiple equilibria.

**Proposition 3** *Suppose that $y_t = y$ is a constant (i.e., $\sigma = 0$ and $\mu = 0$) and the creditors have asynchronous contract periods. There exist $y_h^c > y_l^c > 0$ such that when $y > y_h^c$, it is optimal for an individual creditor to roll over even if the other creditors will all choose to withdraw in the future; when $y < y_l^c$, it is optimal for the creditor to withdraw even if other creditors will all choose to roll over in the future; and when $y \in [y_l^c, y_h^c]$, the creditor's optimal choice depends on others', i.e., it is optimal to withdraw if others will choose to withdraw in the future and it is optimal to roll over if others will choose to roll over.*

Proposition 3 shows that when the bank fundamental is constant over time, multiple equilibria could arise even if creditors make their rollover decisions at different times, in a similar way as in the case when creditors make their decisions at the same time (e.g., Proposition 2). There are also two dominance regions: in the upper (lower) dominance region where the bank fundamental is sufficiently high (low), an individual creditor finds rollover (withdrawal) as the dominant strategy, regardless of the other creditors' choices in the future. However, in the intermediate region, two equilibria emerge–the creditors could all choose either to roll over or to withdraw. Specially, once the fundamental is away from—though arbitrarily close to—the upper or lower dominance region, the creditors' collective rollover or withdrawal strategy is self-fulfilling.[7] For instance, once each individual creditor believes that other maturing creditors will all choose to roll over in the future, rollover is optimal for him now. This "no-future-rollover-risk" belief is actually consistent with the

[7] This result also shows that it is not simply the sequential service constraint imposed by the asynchronous contract periods that drives the unique monotone equilibrium in this paper. Using one type of sequential service constraint, Green and Lin (2003) obtains a unique equilibrium without changing fundamental in a setting when each creditor knows his position, which is different from others', in the service line. In contrast to their model, our modeling choice of random maturity implies that all creditors, in deciding whether to roll over, have the same expected maturity of the new debt contracts. As a result, they perceive themselves to have the same relative position in the line a la Green and Lin (2003).

equilibrium outcome because the bank fundamental is a constant and thus always stays above the lower dominance region.

This logic, however, breaks down if the bank fundamental changes over time and is expected to reach either one of the two dominance regions in the future. The creditors' anticipation of this occurrence stops the self-fulfilling multiple equilibria in the intermediate region, and, instead, allows them to backwardly induce the equilibrium in the intermediate region based on the unique equilibrium outcomes at the two ends of the region as boundary conditions. A unique equilibrium thus arises even in the intermediate region.

Depending on whether the fundamental changes deterministically or stochastically, the exact mechanism is slightly different. It is easier to see the mechanism in the deterministic case, which we discuss first. Suppose that the bank fundamental is deterministic (i.e., $\sigma = 0$) but with a nonzero drift (i.e., $\mu \neq 0$). If $\mu > 0$, the fundamental is improving until the bank asset matures. Knowing that once the fundamental is in the upper dominance region creditors will always choose rollover, each creditor will choose rollover right before the fundamental entering the upper dominance region. This in turn implies that each creditor will also choose rollover even before, and so on, knowing that the bank faces no rollover risk in the future. One counter force is that when the fundamental is below 1, the bank asset could mature during a creditor's contract period and thus generating a loss. Thus, when the fundamental is below a threshold $y_{\mu+} < 1$, it is optimal for each creditor to withdraw, and once the fundamental is above this threshold, it is optimal to roll over.

The reverse reasoning applies to the case if $\mu < 0$, i.e., the fundamental continues to deteriorate until the asset matures.[8] Knowing that once the fundamental is in the lower dominance region creditors will always choose withdrawal, each creditor will choose withdrawal right before the fundamental entering the region. This in turn motivates creditors to withdraw even earlier. This backward induction amplifies the creditors' incentive to withdraw, and thus generating excessive rollover risk to the bank. Rollover is optimal only when the current bank fundamental is sufficiently high, i.e., above a threshold $y_{\mu-} > 1$, so that it provides a creditor with enough cushion against the bank's future rollover risk. The following proposition formally derives this unique equilibrium.

**Proposition 4** *Suppose that the bank fundamental is deterministic with a nonzero drift $\mu$.*

1. *When $\mu > 0$, there is a unique monotone equilibrium, in which each creditor chooses*

[8] This situation is similar in nature to the pessimistic information structure in Acharya, Gale, and Yorulmzer (2009).

*rollover if the bank fundamental is above a threshold $y_{\mu+} < 1$, and withdrawal otherwise.*

2. *When $\mu < 0$, there is a similar unique monotone equilibrium with a threshold $y_{\mu-} > 1$.*

We now discuss the case if the bank fundamental changes randomly over time (i.e., $\sigma > 0$). When the bank fundamental is currently in the intermediate region, it will only move out of the region at a random time in the future. However, backward induction still allows each creditor to pin down a unique equilibrium strategy. Consider the fundamental level right at the boundary of the lower dominance region. At this point, a creditor is indifferent between rollover and withdrawal if the other creditors will always choose rollover in the future. But, knowing that the bank fundamental will stay inside the lower dominance region in the future for a significant portion of time and the maturing creditors will all withdraw inside the lower dominance region, an individual creditor will choose withdrawal at the boundary now. Then, knowing all the future maturing creditors will also update their strategy and choose to withdraw at this level, each creditor will choose withdrawal at an even higher level, and so on. This reasoning is analogous to the rat-race mechanism illustrated in Figure 2. It shows that random shocks can serve the same role as deterministic drifts–allowing the creditors to backwardly induce the equilibrium in the intermediate region based on the unique equilibrium outcomes in the two dominance regions. This is also the insight previously pointed out by Frankel and Pauzner (2000).[9] This mechanism exactly leads to the unique equilibrium derived in Theorem 1.

## 4.5 A Benchmark Model with a Single Creditor

To evaluate the effect of the rollover risk on the bank-run equilibrium, it is useful to establish a benchmark model in which there is only a single creditor holding all the debts of the bank. Because the single creditor no longer needs to worry about the bank's future rollover risk with other creditors, his rollover threshold is free of the coordination problem among creditors.

We keep the basic model structure introduced in Section 3 and simply remove the creditor heterogeneity by assuming that a single creditor holds all the debts of the bank. As before,

[9] Note an important difference between our model and Frankel and Pauzner (2000), which exogenously specify a strict strategic complementarity among agents, i.e., each agent's incentive to take one action increases with the number of agents who take the same action. In contrast, the payoff to the bank creditors in our model is endogenously determined by realistic debt contracts. While the complementarity among creditors exists in most cases, it is violated when the bank fundamental is below the bank's liquidation value. In this case, a creditor who is currently locked in by a contract prefers other creditors to withdraw so that their withdrawals can force the bank to liquidate its asset at a price higher than its fundamental. This strategic substitution effect is also different from the congestion effect in the static bank run model of Goldstein and Pauzner (2005).

the creditor faces a contract period which expires upon the arrival of a Poisson shock with intensity $\delta$. When the contract expires, the creditor can decide whether to roll over the debt for another random contract period or not. If he decides not to roll over, the bank goes bankrupt and has to liquidate its asset. In this event, the creditor's payoff is $\min(L + ly_t, 1)$.

Denote the single creditor's value function as $V^s(y_t)$. We can simply modify the Bellman equation in (5) to get the following one:

$$\rho V^s = \mu y V_y^s + \frac{\sigma^2}{2} y^2 V_{yy}^s + r + \phi [\min(1, y) - V^s] + \delta \max_{\text{rollover or run}} \{\min(L + ly, 1) - V^s, 0\}.$$

Different from the main model, the single creditor does not face the bank's rollover risk with other creditors. Instead, the creditor fully controls whether the bank will fail, which occurs only when the creditor decides not to roll over his debt. As before, the creditor's rollover policy is a function of the asset fundamental $y$, and we again focus on monotone strategies, i.e., to roll over if $y$ is higher than a threshold $y_s$ and to withdraw otherwise.

We summarize the equilibrium in the following theorem:

**Theorem 5** *If there is a single creditor who finances all the debts of the bank. The single creditor's value function $V^s(y)$ is given by the following two cases: if $y_s < 1$,*

$$V^s(y_t) = \begin{cases} \frac{r+\delta L}{\rho+\phi+\delta} + \frac{\phi+\delta l}{\rho+\phi+\delta-\mu} y_t + B_1 y_t^{\eta_3} & \text{when } 0 < y_t < y_s \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t + A_2 y_t^{-\gamma_2} + B_2 y_t^{\eta_2} & \text{when } y_s < y_t < 1 \\ \frac{r+\phi}{\rho+\phi} + A_3 y_t^{-\gamma_2} & \text{when } 1 < y_t \end{cases}$$

*and if $y_s \geq 1$,*

$$V^s(y_t) = \begin{cases} \frac{r+\delta L}{\rho+\phi+\delta} + \frac{\phi+\delta l}{\rho+\phi+\delta-\mu} y_t + D_1 y_t^{\eta_3} & \text{when } 0 < y_t < 1 \\ \frac{r+\phi+\delta L}{\rho+\phi+\delta} + \frac{\delta l}{\rho+\phi+\delta-\mu} y_t + C_2 y_t^{-\gamma_3} + D_2 y_t^{\eta_3} & \text{when } 1 < y_t < y_s \\ \frac{r+\phi}{\rho+\phi} + C_3 y_t^{-\gamma_2} & \text{when } y_s < y_t \end{cases}$$

*where $\eta_2$, $\gamma_2$, $\eta_3$, $\gamma_3$, $A_2$, $A_3$, $B_1$, $B_2$, $C_2$, $C_3$, $D_1$ and $D_2$ are expressions of the model parameters and $y_s$, given in the Appendix A.5. The equilibrium threshold $y_s$ is uniquely determined by the condition that $V^s(y_s) = \min(L + ly_s, 1)$.*

There are two effects affecting the creditor's value function. One is the bank's default risk, i.e., the bank could become insolvent in the future as its fundamental drops below the debt value. Thus, an increase in the bank's fundamental volatility increases the default risk of the bank. In other words, because the creditor's debt payoff is concave with respect to the fundamental, a higher volatility tends to reduce the creditor's value function, especially when the bank's fundamental is near its debt value. The other effect is that once the bank fundamental drops below the debt value, the creditor also has an embedded option of keeping the bank alive. To see this, the creditor's down side is limited by the liquidation value of the bank asset, but he might be able to gain back the full debt value if the bank fundamental improves in the future. Because of the embedded option, an increase in the fundamental volatility or a decrease in the liquidation value of the bank asset would also make the creditor more reluctant to liquidate the bank.

Comparing our main model to the benchmark model with a single creditor, the existence of multiple creditors with asynchronous contract periods introduces an additional source of risk–the bank's future rollover risk with other creditors–to each individual creditor. This rollover risk is the main focus of the following analysis.

# 5 Comparative Statics

In this section, we analyze several comparative statics results of our model. We focus on two model parameters–the liquidation value and volatility of the bank asset. For illustration, we will use a set of baseline values for the model parameters:

$$\rho = 0.05,\ r = 0.10,\ \delta = 30,\ \theta = 1,\ \phi = 1,\ L = 0.85,\ l = 0.05,\ \mu = 5\%,\ \sigma = 10\%. \tag{7}$$

These values imply that the creditors use a rate $\rho = 5\%$ to discount future cashflows; the bank asset pays a stream of interest payment at a rate of 10% per annum to the creditors, which is much higher than the creditors' discount rate; each debt contract has an average duration of about two weeks ($1/\delta$); the bank asset on average lasts for one year ($1/\phi$), which is longer than the debt duration; the asset liquidation value ($L + ly$) is between 85 to 90 cents per dollar when $y$ is between 0 and 1; the asset fundamental $y$, which determines the asset payoff at maturity, has a growth rate of 5% per annum and a volatility of 10% per annum.

## 5.1 Effects of Liquidation Value

The bank's asset liquidation value plays an important role in determining the creditors' rollover threshold. To illustrate the effect of the liquidation value, in Figure 3 we plot the equilibrium threshold $y_*$ against $L$ by varying its value from 0.6 to 0.95 and keeping the

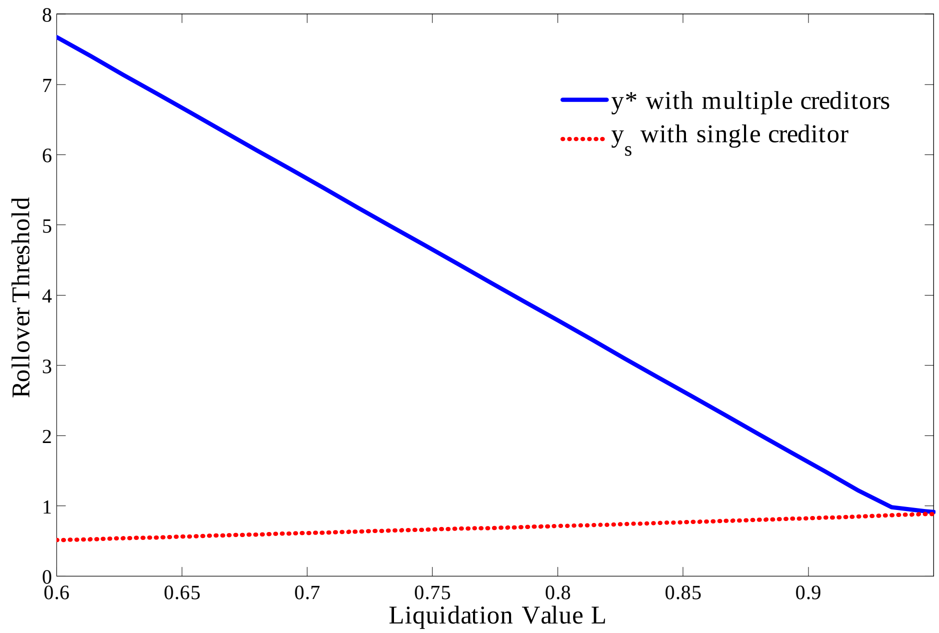


Figure 3: Rollover threshold vs liquidation value

other parameters in (7) the same. The figure also shows the liquidation threshold $y_s$ from the benchmark model with a single creditor.

Figure 3 shows several interesting patterns. First, $y_s$ increases with $L$, while $y_*$ decreases with $L$. To understand this pattern, first consider the benchmark model with a single creditor. As we discussed in Section 4.5, the single creditor has full control over the liquidation of the bank. When he chooses to liquidate the bank, he gets $L$. Thus, as $L$ becomes lower, the creditor is more reluctant to liquidate the bank. In contrast, in the main model with multiple creditors, an individual creditor has no control over the liquidation of the bank. He withdraws to avoid getting $L$ when the bank is forced into a liquidation by other creditors' withdrawals during his contract period. As a result of this negative externality, a lower $L$ motivates each creditor to withdraw even earlier (i.e., choosing a higher rollover threshold) in the multiple-creditor case, while in the single-creditor case, the single creditor internalizes the liquidation loss. We formally prove this result in the following proposition:

**Proposition 6** *When there is a continuum of creditors with asynchronous contract periods, the creditors' equilibrium rollover threshold $y_*$ decreases with the asset liquidation value $L$. In contrast, in the single creditor case, the single creditor's liquidation threshold $y_s$ increases with $L$.*

Since the only difference between our main model and the benchmark model with a single creditor is the bank's rollover risk, the dramatic difference between $y_*$ and $y_s$ demonstrates the significant role played by the rollover risk. It is important to note that the rollover risk ultimately originates from the coordination problem between creditors who makes their rollover decisions at different times. Since all the creditors are risk neutral and identical in other dimensions except their contracts mature at different times, if they were able to coordinate their actions by delegating their rollover decisions to a single agent, we would exactly get the same outcome as the benchmark model with a single creditor.

Figure 3 also illustrates an alarming possibility that $y_*$ could be higher than 1. In other words, despite that in this illustration the bank asset generates a lucrative stream of interest payments to the creditors and that the asset payoff at maturity is expected to be higher than the debt value, the creditors might still choose not to roll over their debts when the asset liquidation value is sufficiently low. The reason is exactly their concerns about the future rollover risk.

Furthermore, the figure shows that the sensitivity of the rollover threshold to the liquidation value is much higher in the region with $y_* \geq 1$ than that in the region with $y_* < 1$. This pattern suggests that if the liquidation value is already lower than a certain level, the creditors' rollover threshold is highly sensitive to any small reduction in the liquidation value. Why is this the case? The source of the high sensitivity in this region originates from the payoff cap of the debt contract at 1, i.e., when the asset fundamental $y$ is already higher than 1, a further improvement of $y$ does not benefit the creditor any more from the payoff at the asset maturity, although it could still increase the creditor's payoff in the event of a forced liquidation from the liquidation value, $L + ly$. Since $l$ is small, this benefit also tends to be small. As a result, in this region with $y_* \geq 1$, the rollover threshold becomes highly sensitive to a small reduction in the liquidation value. We can provide a formal statement of the sensitivity of the rollover threshold to the liquidation value in two limiting cases:

**Proposition 7** *When $y_*$ is very large ($>> 1$), the sensitivity of the threshold to the liquidation value is proportional to $\frac{1}{l}$, a large number. When $y_*$ is very small ($<< 1$), the sensitivity of the threshold to the liquidation value is smaller.*

## 5.2 Effects of Fundamental Volatility

Next, we discuss the effects of the bank asset's fundamental volatility. We again use the parameter values given in (7) as the baseline and then vary the value of $\sigma$. Figure 4 plots

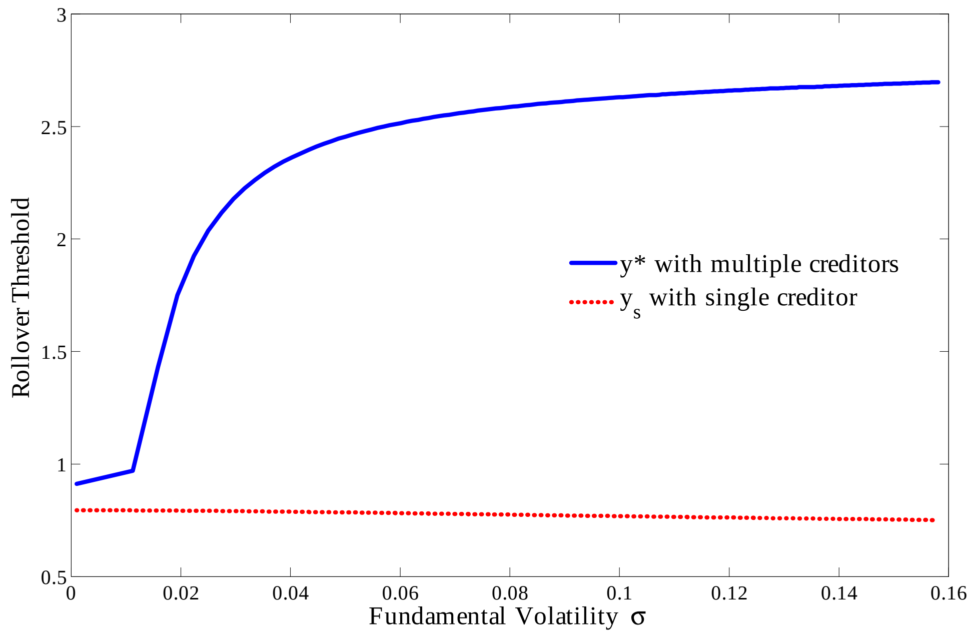


Figure 4: Rollover threshold vs asset fundamental volatility

$y_*$ for the multiple creditor case and $y_s$ for the single creditor case as $\sigma$ increases from slightly above 0 to 16%. Several interesting patterns emerge from this figure. First, as the fundamental volatility increases, $y_s$ has a modest drop from a level around 0.8 to around 0.75, while $y_*$ increases dramatically from 0.9 to 2.7. Again, in the single creditor case, since the creditor has a full control over the bank's liquidation, a larger volatility makes the option value of delaying liquidation more valuable, and therefore motivating a lower rollover threshold. In contrast, in the multiple creditor case, each creditor is exposed to the bank's rollover risk during his lockup period. Instead of providing a higher option value, a larger volatility makes it more likely for the asset fundamental $y$ to hit below other maturing creditors' rollover threshold and potentially leading the bank to fail. Thus, an individual creditor would adopt a higher threshold to safe guard himself for such risk.

An obvious, but important, point from Figure 4 is that even controlling for the asset liquidation value at a reasonably high level 0.85, creditors could start to run on a healthy bank (i.e., with asset fundamental $y$ above the debt value 1) if the fundamental volatility is higher than a certain level (around 1% in the illustration). Figure 4 also shows that once $y_*$ is higher than 1, it is highly sensitive to a change in the fundamental volatility. The high sensitivity of the threshold to a volatility change in this region again originates from the payoff cap of the debt contract at 1. When the asset fundamental $y$ is already higher than 1, a creditor does not gain any more from waiting for a further improvement in the fundamental because his payoff from the asset payoff at maturity is capped at 1. As a result, a small increase in the fundamental volatility can cause a sharp increase in the rollover threshold. Relating back to the ongoing financial crisis, this result shows that the increase in financial market volatility exposes even strong banks with asset fundamental substantially above their debt values to dire bank runs.

## 5.3 Effects of Rollover Frequency

We now discuss the effects of the bank's rollover frequency $\delta$, another key determinant of its rollover risk. As $\delta$ becomes higher, each creditor's contract period, which has an expected length of $1/\delta$, gets shorter. This generates two opposing effects on the equilibrium. First, each individual creditor is locked in for a shorter period. As a result, the creditor has the flexibility to pull out more quickly if the bank's fundamental deteriorates. The increased flexibility makes the creditor more willing to roll over his debt, i.e., to choose a lower rollover threshold. On the other hand, a higher $\delta$ also means that other creditors are locked in for a shorter period. In other words, during the creditor's contract period, the bank is more susceptible to the rollover risk created by other creditors. The increased rollover risk therefore motivates him to choose a higher rollover threshold. The equilibrium threshold $y_*$ trades off the flexibility effect and the rollover risk effect.

Figure 5 plots the rollover threshold as we vary $\delta$ from 0.1 to 60 while keeping other parameters in (7) the same. When $\delta = 0.1$, the rollover threshold $y_* = 0.94$. As $\delta$ increases, $y_*$ increases and hits 1 around $\delta = 2$. Afterwards, $y_*$ increases at an even greater slope, and one can show that $y_* \to \frac{1-L}{l}$ when $\delta$ goes to infinity. The monotonically increasing pattern in $y_*$ with respect to $\delta$ suggests that the rollover risk effect dominates the flexibility effect in this illustration.[10]

Interestingly, Figure 5 also shows that in the single-creditor benchmark case there is an opposite pattern: as $\delta$ increases from 1 to 100, the single creditor's rollover threshold $y_s$ drops slightly from 0.59 to 0.57. This pattern exactly demonstrates that in the absence of the rollover risk, the increased flexibility of pulling out of a troubled bank, resulted from a higher $\delta$, makes the creditor more willing to roll over.

The important role played by the bank's rollover frequency motivates a natural question:

[10] In unreported numerical analysis, we also find that the flexibility effect could dominates the rollover risk effect when $\theta$ is low, i.e., when the bank is robust to the withdrawals by creditors.

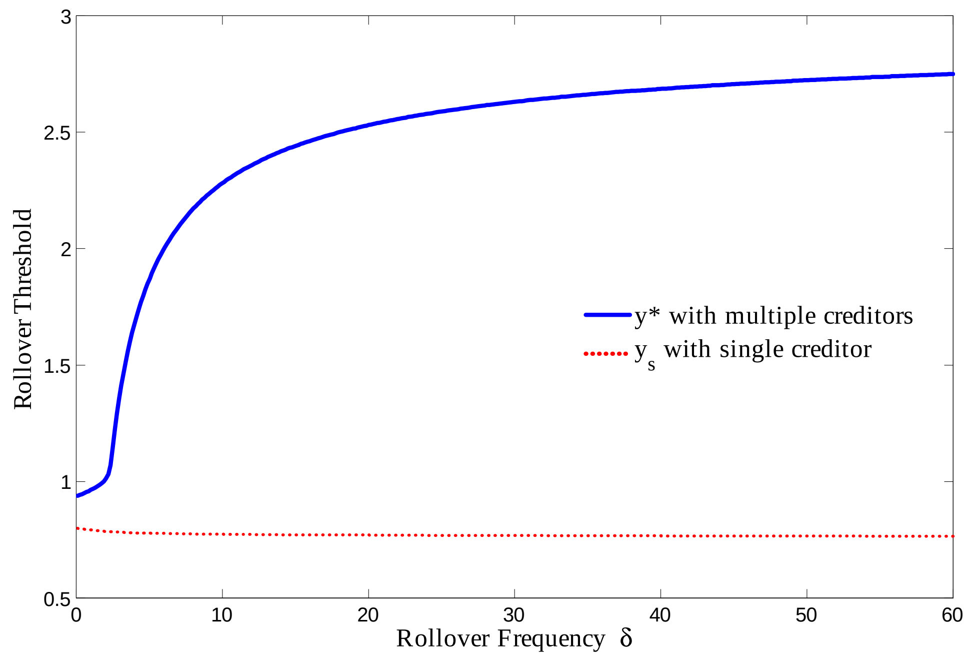


Figure 5: Rollover threshold vs rollover frequency

What would happen if creditors are allowed to choose their rollover frequency? It is intuitive from our earlier discussion that each creditor would prefer a higher rollover frequency for himself so that he has more flexibility to pull out of a troubled bank, while prefer the bank to lock in other creditors for longer period to reduce its rollover risk. More formally, we can derive the following proposition:

**Proposition 8** *Controlling for the other creditors' rollover frequency, each creditor's value function increases with his own rollover frequency.*

This proposition suggests that in the absence of any commitment device like debt covenants or regulatory requirement, both the bank and an atomless individual creditor can gain from reducing the maturity of the new contract without affecting the overall probability of the bank failure. Since this argument applies to every creditor, this maturity resetting process can trigger another rat race between the creditors, in addition to the one illustrated in Section 4.2 in choosing the rollover threshold. As every creditor prefers to have the option to pull out before others when the fundamental is falling, everyone wants a maturity shorter than the others'. As a result, the equilibrium rollover frequency $\delta$ would diverge to infinity, which translates to ultra-short-term financing with zero maturity. This maturity rat race would, however, make the bank highly unstable and thus generate negative externality to other creditors. This mechanism explains why short-term financing becomes more and more pervasive—to some extent overly used—by financial institutions, and thus calls for regulatory measures to force creditors to extend longer term financing in order to stabilize the financial system. For simplicity, we do not explicitly analyze the endogenous interest payments that would arise from this maturity rat race. Brunnermeier and Oehmke (2009) provide a model with two periods to focus on analyzing this issue.

# 6 Discussions

## 6.1 The Financial Crisis of 2007-2008

The Federal Reserve Chairman Ben Bernanke (2008) made the following remark about the mechanism that had led to the financial crisis of 2007-2008:

> *"Since August, mortgage lenders, commercial and investment banks, and structured investment vehicles have experienced great difficulty in rolling over commercial paper backed by subprime and other mortgages. More broadly, a loss of confidence in credit ratings led to a sharp contraction in the asset-backed commercial paper market as short-term investors withdrew their funds... In March, rapidly unfolding events demonstrated that even repo markets could be severely disrupted when investors believe they might need to sell the underlying collateral in illiquid markets. Such forced asset sales can set up a particularly adverse dynamic, in which further substantial price declines fan investor concerns about counterparty credit risk, which then feed back in the form of intensifying funding pressures."*

Our model exactly captures this adverse dynamic generated by creditors' fear of the financial institutions' future funding problems. More precisely, the adverse dynamic arises in our model through the rat race between the creditors in choosing higher and higher rollover thresholds, as illustrated in Section 4.2. Or put differently, if the bank fundamental is not sufficiently high, each creditor takes the earliest possible chance to run. Our model points out two important ingredients in leading to this adverse dynamic. First, during the recent period, largely increased price volatility of many assets held by the financial institutions, such as mortgages, caused serious concerns among the creditors that the bank fundamental might drop below other maturing creditors' rollover threshold and causing them to withdraw funding. Second, the large price discount from liquidating the assets in the severely disrupted and illiquid secondary markets further strengthened the concerns. As a result of these factors, in deciding whether to rollover the debt, each creditor is worried about the severe outcome caused by other creditors' future withdrawals, and would thus choose to withdraw now. This type of preemptive runs by creditors explains the suddenly disappearing debt capacity of many financial institutions (such as Bear Stearns and Lehman Brothers) even though many pundits have augured that their asset fundamentals right before their collapses were still healthy.

Commentators often attribute the funding problems of many financial institutions in the recent period to one of two distinctive factors, either a breakdown in the financial market functioning or fundamental concerns about the institutions' solvency. Our model provides a more complete picture by showing that these two factors working together to create the funding problems (or the rollover risk) of financial institutions. For institutions with strong fundamentals, the capital market functioning is irrelevant. The trouble lies with those institutions whose fundamentals are healthy, but not yet strong enough to fully fence out each creditor's concern about the possible fundamental deterioration during his future contract periods to below the necessary level to ensure funding from other creditors. For these institutions, the severe disruptions in the financial markets for absorbing their possible asset liquidation make it even more difficult for creditors to coordinate their rollover decisions at different times.

The preemptive runs by creditors also lie at the heart of the challenges confronting the governments and central banks' efforts in restoring the world financial system. To fix this central problem, our model suggests removing two ingredients that led to the problem should be the focus of any government policy efforts. One type of policies is for the central bank to improve the liquidation value of the assets. By reducing the potential loss from a future forced liquidation, such a policy can mitigate the incentive for the creditors' preemptive runs. This argument thus provides a rationale for the wide range of lending facilities created by the Federal Reserve in the recent period to boost the market liquidity. A good example is the Federal Reserve facility to buy high-quality commercial paper at a term of three months. Following a prominent money market mutual fund's "breaking the buck" (i.e., a decline of its net asset below par) in September 2008, investors started to withdraw money in large amounts from money market funds that invest in commercial paper. Created right at this time, the Federal Reserve facility provides a backstop on the funds' liquidation value of their commercial paper (i.e., a guarantee on $L$ in our model). By soothing investors' concerns about the money market funds' future funding problems, this facility has been successful in preventing the adverse dynamic of investors trying to be the earlier ones to withdraw from the funds.

Another type of policies is to segregate the deteriorating assets held by the troubled financial institutions. As recently remarked by the Federal Reserve Chairman Ben Bernanke (2009),

> "*a continuing barrier to private investment in financial institutions is the large quantity of troubled, hard-to-value assets that remain on institutions' balance sheets. The presence of these assets significantly increases uncertainty about the underlying value of these institutions and may inhibit both new private investment and new lending.*"

From the perspective of our model, taking the troubled assets out of the institutions balance sheets serves two purposes. First, it reduces the fundamental volatility $\sigma$, thus reducing the institutions' future rollover risk. Second, by providing a fixed payment for the hard-to-value assets, it also protects the assets' liquidation value $L$ from drifting further down with the market strains, which again mitigates the institutions' future rollover risk. Taken together, our model also justifies the initial Troubled Asset Relief Program (TARP) proposed by the former Treasury Secretary Henry Paulson.

## 6.2 Evaluating the TARP Program

Our model provides a simple and reasonably tractable framework to quantitatively evaluate the government policies. As an illustration, we analyze the effect of the TARP program on a hypothetical bank, which has the asset and debt structures described in Section 3 and with the parameter values given in (7).

We can view the bank as having a portfolio of two assets: a riskless asset and a risky asset. The constant payout $r$ represents the riskless (safe and good) asset; while the random maturity payoff represents the risky (volatile and potentially troubled) asset that matures in the future. Consider a hypothetical policy in the spirit of the TARP program, in which the government exchanges a fraction $\alpha$ of the risky asset for a fixed payment to the bank at the time when the bank asset matures or when the bank is forced into liquidation. There are many different ways to determine an appropriate price for the risky asset, which generates

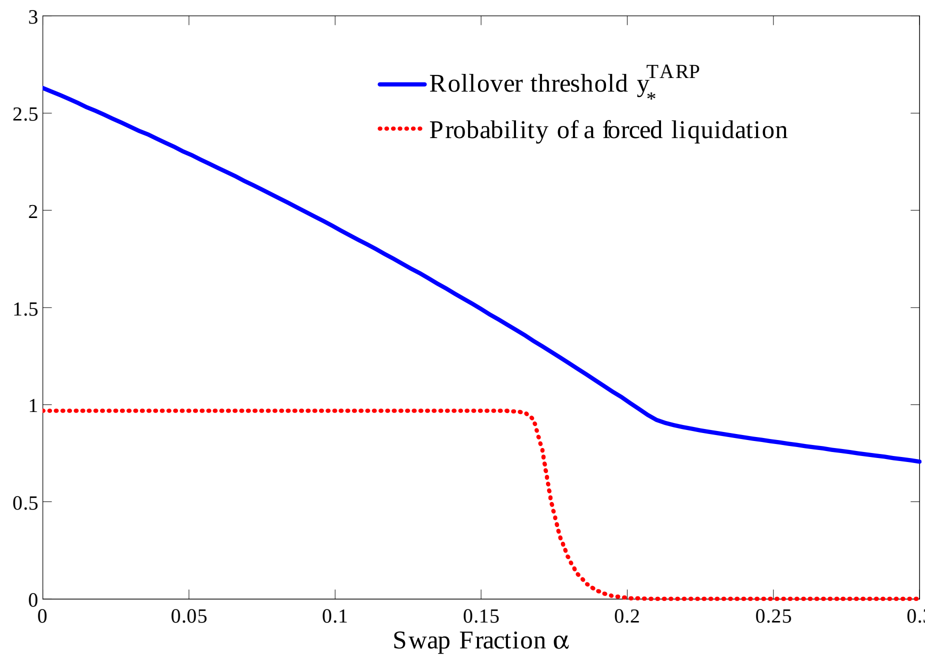


Figure 6: Evaluating the TARP program

at cashflow of $y_{\tau_\phi}$ at a random maturity $\tau_\phi$. Since the purpose of our illustration is simply to show how effectively such a swap program can reduce the creditors' runs on the bank, we simplify the pricing policy and set the price to be the current fundamental level $y_0$. Given this price, the swap program changes the liquidation value of the bank asset to be

$$L' = \alpha y_0 + (1 - \alpha) L \text{ and } l' = (1 - \alpha) l,$$

and the creditors' payoff at the time when the bank asset matures to be

$$\min \left\{ \alpha y_0 + (1 - \alpha) y_{\tau_\phi}, 1 \right\}.$$

Then, we can solve the bank-run equilibrium with these new payoffs to the creditors and obtain the new equilibrium rollover threshold $y_*^{TARP}$. Based on this threshold and the initial fundamental level $y_0$, we also calculate the probability that the bank will ever be forced into liquidation before its asset matures, i.e., $\Pr(\tau_\theta > \tau_\phi)$, which is a reasonable measure of the stability of the bank.

In Figure 6, we plot the new equilibrium threshold $y_*^{TARP}$ and the probability of a forced bank liquidation with respect to the fraction of the risky asset swapped by the government. In this illustration, we take the initial fundamental $y_0$ to be 1.3, and the corresponding equilibrium rollover threshold of creditors is around 2.6, which is substantially above the current fundamental level. As a result, the bank is already in grave danger of a forced liquidation with a probability of 97%. The figure shows that as the government's swap fraction goes up from 0 to 0.3, the creditors' rollover threshold drops monotonically from 2.6 (the status quo) to a level below 0.7. The drop in the threshold is especially rapid before it hits 1. The reason as we explained before is due to payoff cap of the debt contract at 1. The figure also shows that the TARP program can effectively reduce the probability of a forced bank liquidation. After the government swaps out 20% of the bank's risky asset, the probability of a forced liquidation drops to almost zero. Furthermore, if the swap fraction is lower than 17%, its impact on the liquidation probability is negligible, but once the fraction rises to around 18%, the liquidation probability quick drops down. This is because that at this level, the creditors' equilibrium rollover threshold is now below the current fundamental level.

## 6.3 Credit Risk Modeling

Credit risk is an increasingly important factor in determining firms' cost of capital. The standard credit risk modeling approach, following the classic structural models of Merton (1974) and Leland (1994), determines a firm's credit risk by the probability of and the loss from the event that the firm's asset value drops below the debt value or that the firm's equity drops to zero. This approach, however, ignores an important course of creditor risk–the firm's rollover risk. Like banks, industrial firms also rely on a large pool of public investors to fund their debt financing and need to regularly raise new debts to retire maturing debts. As a result, they are also affected by the potential coordination problem between different investors in the form of the rollover risk modeled in this paper. According to our model, investors' anticipation of a firm's difficulty to roll over its debts in the future, for reasons related to either the firm's fundamental or the capital market liquidity, could already affect the firm's ability to raise capital now. More precisely, the inability of investors to perfectly coordinate their rollover decisions leads to a higher default threshold in the firm's asset value than what the standard models imply. Thus, by incorporating the rollover risk, our model can help resolve a challenge faced by the these standard structural models in explaining the large credit spreads observed in the data.

In this section, we provide an example to illustrate the importance of the rollover risk in credit risk modeling. Consider two banks (which could also be broadly interpreted as two industrial firms), one with the same asset and debt structure as described in our main model with a continuum of creditors, while the other as described in the benchmark model with a single creditor. These two banks are otherwise identical except the composition of creditors. Suppose that the two banks' debt contracts all expire at the same rate of $\delta$, i.e., the banks need to roll over their debts at a rate of $\delta$.

We would like to evaluate the prices of two hypothetical zero coupon bonds with face value 1 and maturity $T$ issued by these two banks. Suppose that each bond provides the following payoff depending on the following three scenarios: 1) if the bank's asset matures before $T$ and a forced liquidation, the bond pays $\min\left(y_{\tau_\phi}, 1\right)$; 2) if a forced liquidation occurs before $T$ and the asset maturity, the bond pays $L + ly_{\tau_\theta}$, the liquidation value of the bank asset; 3) otherwise, the bond pays 1. This payoff effectively captures the bank's credit risk before time $T$. We also assume that this bond has an infinitesimal amount on the bank's balance sheet so that it does not affect the bank's debt structure. One can also think this bond as a derivative contract traded in the secondary markets (like a credit swap). Since investors are risk neutral, the time-0 price of this bond $P$ is simply the expected value of the bond payoff discounted by a rate $\rho$. The implied bond yield is $\beta = -\frac{\ln P}{T}$. As commonly used in practice, we measure the bond's credit risk by its credit spread, i.e., the spread between its yield and the yield of a risk-free bond with the same maturity.[11]

Figure 7 plots the credit spread of these two bonds with respect to the two banks' debt rollover frequency $\delta$, based on the model parameters given in (7) and $y_0 = 1$, $T = 0.25$. The difference between these two credit spreads measures the contribution of bank-run risk to the credit risk of the bank with multiple creditors. As the rollover frequency $\delta$ increases from 25 to 50 (i.e., from once every two weeks to once every week), the difference between the two credit spreads rises sharply from almost zero to more than 25%.

This illustration suggests that rollover risk can be a substantial part of the credit risk of a firm that features multiple creditors and a short-term debt structure. One testable implication is that the rollover risk decreases with the debt maturity.

[11] Our risky bond receives a payoff at a random time before the bond maturity $T$. For a fair comparison, we also impose the same random maturity on the risk-free bond, which has a value of $\frac{\phi}{\rho+\phi} + \frac{\rho}{\rho+\phi}e^{-(\rho+\phi)T}$. Then we calculate the yield earned by the risk-free bond as $\beta_{riskfree} = -\frac{1}{T}\ln\left(\frac{\phi}{\rho+\phi} + \frac{\rho}{\rho+\phi}e^{-(\rho+\phi)T}\right)$. The credit spread is measured relative to this yield.

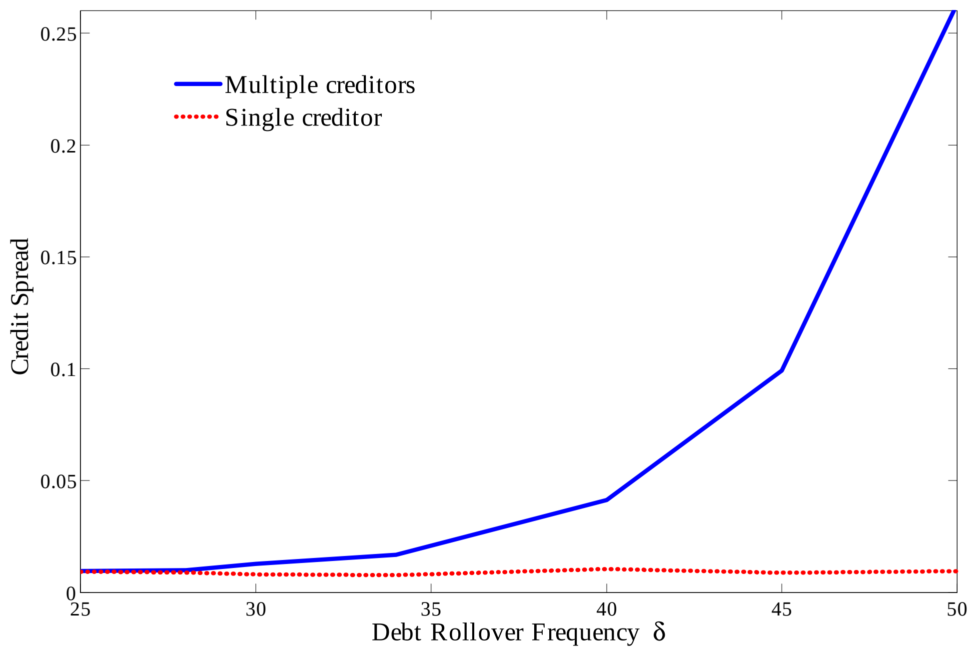


Figure 7: Credit spread vs debt rollover frequency

# 7 Conclusion and Further Discussions

In this paper, we provide a dynamic model of bank runs. We emphasize two important features of a modern bank's debt financing: 1) a duration mismatch between the bank's long-term investment position and short-term debt financing, and 2) multiple creditors with asynchronous contract periods. Different from the static bank-run models with multiple equilibria, we derive a unique monotone equilibrium, in which creditors coordinate their asynchronous rollover actions based on observable fundamental shocks. In deciding whether to roll over the debt, each creditor faces the future rollover risk, i.e., during his contract period falling bank fundamentals may trigger other maturing creditors to run on the bank. More importantly, that a creditor uses a high rollover threshold as a protection against the bank's future rollover risk will motivates other creditors to use an even higher threshold. This rat race mechanism makes the equilibrium threshold substantially higher than the reasonable fundamental needed to justify the solvency of the bank, and, furthermore, to be highly sensitive to the volatility and liquidation value of the bank asset. Thus, our model captures a central element of the ongoing financial crisis–even in the absence of any fundamental deterioration, small changes in the volatility and liquidation value of the bank asset could trigger creditors to run on a solvent bank.

For simplicity, our model ignores several potentially important features of modern banks. In reality, commercial banks typically hold cash reserves against depositors' withdrawal, and investment banks and other financial institutions have liquid asset holdings against normal investment losses and/or investor redemptions. These liquid holdings can buffer some liquidity shocks, though typically not large enough to accommodate the withdrawal of the institutions' entire short-term funding. Even though our model does not explicitly incorporate a cash reserve, the bank's ability to sustain the creditors' withdrawal for a period of time, which is inversely measured by the parameter $\theta$, partially captures the role of a cash reserve inside the bank. The fact that in reality financial institutions usually do not hold a sufficient amount of cash reserves against their short-term liabilities suggests a high opportunity cost of holding cash and/or liquid assets, which supports our simplified treatment. If we incorporate a cash reserve into the model, individual creditors' rollover decision would become reserve dependent, because the reserve becomes part of the bank's liquidation value. We do not expect such an extension to alter the key bank-run mechanism illustrated in the current model, although the reserve could mitigate the bank's rollover risk if it is cheap enough.

Another interesting issue is that as the fundamental deteriorates, the bank could raise interest payments to offset the creditors' incentive to run. However, to do so, the bank needs to have sufficient cash reserves to pay for the increased interest payments, which might not be realistic for a bank in the middle of a bank run. But, nevertheless, this consideration again raises the strategic importance of cash reserves. The bank could choose low interest payments and save some cashflows in normal periods when the fundamental is high, only to pay for the high interest payments in crisis times. We will leave this important and realistic issue for our future research.

# A Appendix

## A.1 Proof of Theorem 1

Using the Bellman equation in (6), we first construct an individual creditor's value function by assuming that he and all the other creditors use the same monotone strategy with a threshold $y_*$. This assumption implicitly imposes that $V(y;y_*) > 1$ for $y > 1$ and $V(y;y_*) < 1$ for $y < 1$. We will verify that this condition indeed holds in the equilibrium later. Under this assumption, the Bellman equation (6) becomes

- If $y < y_*$,

$$0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - [\rho + \phi + (\theta + 1)\delta] V(y;y_*) + \phi \min(1,y) + \theta\delta(L + ly) + r + \delta; \quad (8)$$

- If $y \geq y_*$,

$$0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - (\rho + \phi) V(y;y_*) + \phi \min(1,y) + r. \quad (9)$$

The value function has to satisfy these two differential equations and be continuous and differentiable at the boundary point $y_*$. In solving these differential equations, we need to use the two solutions to the fundamental equation:

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - [\rho + \phi + (1+\theta)\delta] = 0,$$

which are

$$-\gamma_1 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2[\rho + \phi + (1+\theta)\delta]}}{\sigma^2} < 0$$

and

$$\eta_1 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2[\rho + \phi + (1+\theta)\delta]}}{\sigma^2} > 1,$$

and the two solutions to the fundamental equation:

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - (\rho + \phi) = 0, \quad (10)$$

which are

$$-\gamma_2 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2(\rho + \phi)}}{\sigma^2} < 0$$

and

$$\eta_2 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2(\rho + \phi)}}{\sigma^2} > 1.$$

We summarize the constructed value function below.

**Lemma 9** *Suppose that every creditor uses a monotone strategy with a renewal threshold $y_*$. Then the value function of an individual creditor is given by the following two cases:*

- *If $y_* < 1$,*

$$V(y;y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y + B_1 y^{\eta_1} & \textit{when } 0 < y < y_* \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu}y + A_2 y^{-\gamma_2} + B_2 y^{\eta_2} & \textit{when } y_* < y < 1 \\ \frac{r+\phi}{\rho+\phi} + A_3 y^{-\gamma_2} & \textit{when } 1 < y \end{cases} \tag{11}$$

*The four coefficients $B_1$, $B_2$, $A_2$, and $A_3$ are given by*

$$\begin{aligned} B_1 &= \frac{[K_1\gamma_2 + H_1] - y_*^{-\eta_2}(\gamma_2 K_2 + H_2 y_*)}{(\eta_1+\gamma_2)\, y_*^{\eta_1-\eta_2}} \\ B_2 &= \frac{y_*^{-\eta_2}}{\eta_2+\gamma_2}\left[\gamma_2 K_2 + H_2 y_* + B_1(\eta_1+\gamma_2)\, y_*^{\eta_1}\right] \\ &= \frac{1}{\eta_2+\gamma_2}[K_1\gamma_2 + H_1] \\ A_2 &= \frac{y^{*\gamma_2}}{\eta_2+\gamma_2}\left[\eta_2 K_2 - H_2 y_* + B_1(\eta_2-\eta_1)\, y_*^{\eta_1}\right] \\ A_3 &= A_2 - \frac{1}{\eta_2+\gamma_2}[K_1\eta_2 - H_1] \end{aligned}$$

*where*

$$\begin{aligned} K_2 &= \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} - \frac{r}{\rho+\phi} + H_2 y_* \\ K_1 &= -\frac{\phi\mu}{(\rho+\phi)(\rho+\phi-\mu)} \\ H_1 &= -\frac{\phi}{\rho+\phi-\mu} \\ H_2 &= \frac{\theta\delta l(\rho+\phi-\mu) - \phi(1+\theta)\delta}{(\rho+\phi+(1+\theta)\delta-\mu)(\rho+\phi-\mu)} < 0 \end{aligned}$$

- *If $y_* \geq 1$,*

$$V(y;y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y + D_1 y^{\eta_1} & \textit{when } y < 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y + C_2 y^{-\gamma_1} + D_2 y^{\eta_1} & \textit{when } 1 < y < y_* \\ \frac{r+\phi}{\rho+\phi} + C_3 y^{-\gamma_2} & \textit{when } y_* < y \end{cases}. \tag{12}$$

*The four coefficients $D_1$, $D_2$, $C_2$, and $C_3$ are given by*

$$
\begin{aligned}
C_2 &= \frac{M_2\eta_1 - M_1}{(\eta_1+\gamma_1)} < 0 \\
C_3 &= \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2} C_2 y_*^{\gamma_2-\gamma_1} + \frac{\eta_1 - 1}{\eta_1+\gamma_2} \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_*^{\gamma_2+1} \\
&\quad - \frac{\eta_1}{\eta_1+\gamma_2}\left[\frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}\right] y_*^{\gamma_2} \\
&= \frac{(\eta_1+\gamma_1) C_2 (y_*)^{-\gamma_1} - \eta_1 N_1 - \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_*}{(\eta_1+\gamma_2) y_*^{-\gamma_2}} \\
D_2 &= \frac{(\gamma_1-\gamma_2) C_2 (y_*)^{-\gamma_1} + \gamma_2 N_1 - \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_*}{(\eta_1+\gamma_2) y_*^{\eta_1}} \\
D_1 &= D_2 - \frac{M_2\gamma_1 + M_1}{(\eta_1+\gamma_1)}
\end{aligned}
$$

*where*

$$
\begin{aligned}
M_1 &= \frac{\phi}{\rho+\phi+(1+\theta)\delta-\mu} \\
M_2 &= \frac{\phi\mu}{(\rho+\phi+(1+\theta)\delta)(\rho+\phi+(1+\theta)\delta-\mu)} \\
N_1 &= \frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} - \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_*
\end{aligned}
$$

**Proof.** *We first consider the case that $y_* < 1$.*

Depending on the value of $y$, we have the following three scenarios.

- If $0 < y < y_*$ :

$$\frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - [\rho+\phi+(1+\theta)\delta] V(y) + (\phi+\theta\delta l) y + r + \theta\delta L + \delta = 0.$$

  The general solution of this differential equation is given in the first line of equation (11) with the coefficient $B_1$ to be determined by the boundary conditions. Note that to ensure the value of $V$ to be finite as $y$ approaches zero, we have ruled out another power solution of the equation $y^{-\gamma_1}$.

- If $y_* < y < 1$ :

$$\frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - (\rho+\phi) V(y) + \phi y + r = 0.$$

  The general solution of this differential equation is given in the second line of equation (11) with the coefficients $A_2$ and $B_2$ to be determined by the boundary conditions.

- If $y > 1$ :

$$\frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - (\rho + \phi) V(y) + r + \phi = 0.$$

The general solution of this differential equation is given in the third line of equation (11) with the coefficient $A_3$ to be determined by the boundary conditions. Note that to ensure the value of $V$ to be finite as $y$ approaches infinity, we have ruled out another power solution of the equation $y^{\eta_2}$.

To determine the four coefficients $B_1$, $B_2$, $A_2$, and $A_3$, we have four boundary conditions at $y = y_*$ and 1, i.e., the value function $V(y)$ must be continuous and differentiable at these two points. Solving these boundary conditions leads to the coefficient values given in Proposition 9.

*We next consider the case that* $y_* \geq 1$.

Depending on the value of $y$, we have the following three scenarios.

- If $0 < y < 1$:

$$\frac{1}{2} \sigma^2 y^2 V_{yy} + \mu y V_y - [\rho + \phi + (1 + \theta) \delta] V(y) + (\phi + \theta \delta l) y + r + \theta \delta L + \delta = 0.$$

The general solution of this differential equation is given in the first line of equation (12) with the coefficient $D_1$ to be determined by the boundary conditions. Note that to ensure the value of $V$ to be finite as $y$ approaches zero, we have ruled out another power solution of the equation $y^{-\gamma_1}$.

- If $1 < y < y_*$:

$$\frac{1}{2} \sigma^2 y^2 V_{yy} + \mu y V_y - [\rho + \phi + (1 + \theta) \delta] V(y) + \theta \delta l y + r + \phi + \theta \delta L + \delta = 0.$$

The general solution of this differential equation is given in the second line of equation (12) with the coefficients $C_2$ and $D_2$ to be determined by the boundary conditions.

- If $y_* < y$:

$$\frac{1}{2} \sigma^2 y^2 V_{yy} + \mu y V_y - (\rho + \phi) V(y) + r + \phi = 0.$$

The general solution of this differential equation is given in the third line of equation (12) with the coefficient $C_3$ to be determined by the boundary conditions. Note that to ensure the value of $V$ to be finite as $y$ approaches infinity, we have ruled out another power solution of the equation $y^{\eta_2}$.

Like in the previous case, to determine the four coefficients $D_1$, $D_2$, $C_2$, and $C_3$, we have four boundary conditions at $y = y_*$ and 1, i.e., the value function $V(y)$ must be continuous and differentiable at these two points. Solving these boundary conditions leads to the coefficient values given in Proposition 9. ■

Based on the value function derived in Lemma 9, we can show that there exists a unique threshold $y_*$ for the equilibrium condition to hold.

**Lemma 10** *There exists a unique $y_*$ such that*

$$V(y_*; y_*) = 1.$$

**Proof.** Define

$$W(y) \equiv V(y; y).$$

We need to show that there is a unique $y_*$ such that $W(y_*) = 1$.

We first show that $W(y)$ is monotonically increasing when $y < 1$. In this case, we can directly extract the value of $W(y)$ from equation (11), which, by neglecting terms independent of $y$, is given

$$\begin{aligned} W(y) &= \left[\frac{\phi + \theta\delta l}{\rho + \phi + (1+\theta)\delta - \mu} - \frac{1+\gamma_2}{\eta_1 + \gamma_2} H_2\right] y + \frac{[K_1\gamma_2 + H_1]}{\eta_1 + \gamma_2} y^{\eta_2} \\ &= \left[-H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2\right] y + \frac{[K_1\gamma_2 + H_1]}{\eta_1 + \gamma_2} y^{\eta_2}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{dW(y)}{dy} &= -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{[K_1\gamma_2 + H_1]}{\eta_1 + \gamma_2} \eta_2 y^{\eta_2 - 1} \\ &> -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{[K_1\gamma_2 + H_1]}{\eta_1 + \gamma_2} \eta_2 \\ &= -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{\eta_2}{\eta_1 + \gamma_2} H_1 + \frac{\gamma_2\eta_2}{(\eta_1 + \gamma_2)} K_1 \\ &= \frac{\eta_1 - 1}{\eta_1 + \gamma_2} (H_2 - H_1) + \frac{\eta_2 - \gamma_2 - 1}{\eta_1 + \gamma_2} H_1 + \frac{\gamma_2\eta_2}{(\eta_1 + \gamma_2)} K_1 \end{aligned}$$

where the inequality is due to $K_1 < 0$ and $H_1 < 0$.

Note that in the first term above,

$$H_2 - H_1 = \frac{\theta\delta l + \phi}{\rho + \phi + (1+\theta)\delta - \mu}$$

is positive according to the parameter restriction in (2). For the second term, note that $\eta_2 - \gamma_2 - 1 = -2\frac{\mu}{\sigma^2}$. Then after some algebraic substitution (note that $\gamma_2\eta_2 = \frac{2(\rho+\phi)}{\sigma^2}$), the sum of the second and third terms is

$$-2\frac{\mu}{\sigma^2}\frac{1}{(\eta_1+\gamma_2)}H_1+\frac{\gamma_2\eta_2}{(\eta_1+\gamma_2)}K_1=0$$

Thus, $\frac{dW(y)}{dy}>0$.

Next, we need to ensure that $W(0)<1$. Equation (11) implies that

$$W(0)=\frac{\eta_1}{(\eta_1+\gamma_2)}\frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}+\frac{\gamma_2}{(\eta_1+\gamma_2)}\frac{r}{\rho+\phi}.$$

The parameter restriction in (1) insures that

$$\frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}<1 \quad \text{and} \quad \frac{r}{\rho+\phi}<1\ .$$

thus, $W(0)<1$.

We now show that $W(y)$ is monotonically increasing when $y\geq 1$. Equation (12) implies that

$$\begin{aligned} W(y) &= \frac{r+\phi}{\rho+\phi}+C_3y^{-\gamma_2} \\ &= \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2}C_2y^{-\gamma_1}+\frac{\eta_1-1}{\eta_1+\gamma_2}\frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y \\ &\quad +\frac{\gamma_2}{(\eta_1+\gamma_2)}\frac{r+\phi}{\rho+\phi}+\frac{\eta_1}{\eta_1+\gamma_2}\frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} \end{aligned}$$

We now show $C_2<0$, which is equivalent to showing $\eta_1<\frac{M_1}{M_2}=\frac{\rho+\phi+(1+\theta)\delta}{\mu}$. Plugging $x=\frac{\rho+\phi+(1+\theta)\delta}{\mu}$ into the fundamental equation, we find that the value is positive. This implies that $\eta_1<\frac{M_1}{M_2}$. Now because $\eta_1>1$, $W(y)$ is increasing in $y$. Also note that $W(\infty)>1$.

Since $W(y)$ is monotonically increasing with $W(0)<1$ and $W(\infty)>1$, there exists a unique $y_*$ such that $W(y_*)=1$. ■

Lemma 10 implies that there can be at most one symmetric monotone equilibrium. Finally, we need to verify that a monotone strategy with the threshold level determined in Lemma 10 is indeed optimal for an individual creditor if every other creditor uses this threshold.

**Lemma 11** *If every other creditor uses a monotone strategy with a threshold $y_*$ identified in Proposition 10, then the same strategy is also optimal for an individual creditor.*

**Proof.** If every other creditor uses the monotone strategy with the threshold $y_*$, to show that the value function constructed from solving the differential equations (8) and (9) is indeed optimal for an individual creditor, we simply need to verify that $V(y) < 1$ for any $y < 1$, and $V(y) > 1$ for any $y > 1$, which directly implies that the value function solves the Bellman equation (6). By construction in Proposition 9, $V(0) = \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} < 1$ and $V(\infty) = \frac{r+\phi}{\rho+\phi} > 1$. We just need to show that $V(y)$ only crosses 1 once at $y_*$.

*We first consider the case that $y_* < 1$.*

We prove by contradiction. Suppose that $V(y)$ also crosses 1 at another point below $y_*$. Then, there exists $y_1 < y_*$ such that

$$V(y_1) > V(y_*) = 1,\ V'(y_1) = 0, \text{ and } V''(y_1) < 0.$$

Using the differential equation (8), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + \phi \min(1, y_1) + \theta\delta(L + l y_1) + r + \delta}{\rho + \phi + (\theta+1)\delta} \\ &< \frac{(\phi + \theta\delta l) y_1 + \theta\delta L + r + \delta}{\rho + \phi + (\theta+1)\delta} < \frac{\phi + \theta\delta l + \theta\delta L + r + \delta}{\rho + \phi + (\theta+1)\delta} < 1. \end{aligned}$$

The last inequality is implied by the parameter restrictions in (1) and (4). This is a contradiction with $V(y_1) > 1$. Thus, $V(y)$ cannot cross 1 at any $y$ below $y_*$. This also implies that $V'(y_*) > 0$.

Next, we show that $V(y)$ is monotonic in the region $y > y_*$. Suppose that $V(y)$ is non-monotone, then there exist two points $y_1 < y_2$ such that

$$V(y_1) > V(y_2),\ V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2)$$

According to the different equation (9), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1)}{\rho + \phi} \\ &> \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2)}{\rho + \phi} = V(y_2) \end{aligned}$$

which is a contradiction.

*We next consider the case that $y_* \geq 1$.*

The argument is similar to the first case. The expression in equation (12) implies that $V(y)$ has to approach $\frac{r+\phi}{\rho+\phi}$ from below (otherwise $V(y_*)$ cannot be below V($\infty$)), thus $C_3 < 0$. This implies that $V(y)$ is increasing on $[y_*, \infty)$, and

$$V'(y_*) > 0.$$

Now consider the region $[0, y]$, it is easy to check that $V'(0) > 0$. Therefore, if $V(y)$ is not monotonic in $[0, y_*)$, there must exist two points $y_1 < y_2$ such that

$$V(y_1) > V(y_2), \; V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2)$$

According to the Bellman equation, we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1) + \delta\left[1 + \theta(L + l y_1)\right]}{\rho + \phi + (1+\theta)\delta} \\ &< \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2) + \delta\left[1 + \theta(L + l y_2)\right]}{\rho + \phi + (1+\theta)\delta} = V(y_2) \end{aligned}$$

which is a contradiction. Thus, $V(y)$ is also monotonically increasing in $[0, y_*)$.

To summarize, we have shown that $V(y)$ only crosses 1 once at $y_*$. Thus, it is optimal for an individual creditor to renew his contract if $y > 1$ and withdraw if $y < 1$.

Sufficient conditions for $L + l y_* < 0$. Fixing $L$, choosing small $l$ such that $l < 1 - L$. Then for $L + l y_* > 0$ we have

$$y_* > \frac{1-L}{l} > 1$$

With this,

$$\begin{aligned} V(y_*) &= \frac{r+\phi}{\rho+\phi} + C_3 y_*^{-\gamma_2} \\ &= \frac{r+\phi}{\rho+\phi} + \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2} C_2 y_*^{-\gamma_1} \\ &\quad + \frac{\eta_1 - 1}{\eta_1+\gamma_2} \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta - \mu} y_* - \frac{\eta_1}{\eta_1+\gamma_2}\left[\frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}\right] \end{aligned}$$

We want to find sufficient conditions that $V(y_*) > 1$, which leads to a contradiction. Plugging in $C_2 = \frac{M_2\eta_1 - M_1}{(\eta_1+\gamma_1)}$, It is equivalent to

$$\frac{r-\rho}{\rho+\phi}\gamma_2 + (M_2\eta_1 - M_1) y_*^{-\gamma_1} + (\eta_1 - 1)\frac{\theta\delta l y_*}{\rho+\phi+(1+\theta)\delta-\mu} + \eta_1 \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} > \eta_1$$

When $l \to 0$, $y_* > \frac{1-L}{l} \to \infty$, so the second term is negligible. Therefore a sufficient condition is

$$\frac{r-\rho}{\rho+\phi}\gamma_2 + (\eta_1 - 1)\left(\frac{\theta\delta l y_*}{\rho+\phi+(1+\theta)\delta} + \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}\right) + \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} > \eta_1$$

Notice that

$$\begin{aligned} &(\eta_1 - 1)\left(\frac{\theta\delta l y_*}{\rho+\phi+(1+\theta)\delta-\mu} + \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}\right) \\ > \; &(\eta_1 - 1)\left(\frac{r+\phi+\theta\delta(L+l y_*)+\delta}{\rho+\phi+(1+\theta)\delta}\right) > (\eta_1 - 1)\left(\frac{r+\phi+(1+\theta)\delta}{\rho+\phi+(1+\theta)\delta}\right) \end{aligned}$$

which implies a condition

$$\frac{r-\rho}{\rho+\phi}\gamma_2+\frac{r-\rho}{\rho+\phi+(1+\theta)\delta}\eta_1>\frac{\theta\delta(1-L)}{\rho+\phi+(1+\theta)\delta}$$

This condition holds for all the parameters that we use in this paper. ■

## A.2 Proof of Proposition 2

We first derive an individual creditor's value function $U(y)$ if the bank survives the creditors' rollover decisions at time 0 and thus will be able to stay until the asset maturity at $\tau_\phi$. $U(y)$ satisfies the following differential equation:

$$\rho U=\mu y U_y+\frac{1}{2}\sigma^2 y^2 U_{yy}+\phi\left[\min(1,y)-U\right]+r.$$

It is direct to solve this differential equation and show that $U(y)$ is a monotonically increasing function with

$$U(0)=\frac{r}{r+\phi}<1 \quad \text{and} \quad U(\infty)=\frac{r+\phi}{\rho+\phi}>1.$$

Then intermediate value theorem implies that there exists $y_l>0$ such that

$$U(y_l)=1.$$

The parameter restriction in (3) also ensures that $y_l<y_h=\frac{1-L}{l}$, a very large number.

Next, we show that if $y_0>y_h$, then it is optimal for an individual creditor to roll over even if all the other creditors choose to withdraw. In this case, there is a probability of $\theta$ that the bank cannot find new creditors to replace the outgoing ones and is forced into a premature liquidation. But the liquidation value is sufficient to pay off all the creditors because $L+ly_0>1$. Thus, the creditor's expected payoff from choosing withdrawal is

$$\theta+(1-\theta),$$

while his expected payoff from choosing rollover is

$$\theta+(1-\theta)\,U(y_0),$$

which is higher than the expected payoff from choosing withdrawal.

Next, we show that if $y_0<y_l$, then it is optimal for an individual creditor to withdraw even if all the other creditors choose to roll over. In this case, the bank will always survive no matter what the single creditor's decision is. If he chooses to withdraw, he gets a payoff

of 1, while if he chooses to roll over, his continuation value function is $U(y_0)$. Thus, it is optimal for the creditor to withdraw.

Finally, we consider the case when $y_0 \in [y_l, y_h]$. If all the other creditors choose to roll over, then an individual creditor's payoff from withdrawal is 1, while his continuation value function is $U(y_0)$. Thus it is optimal for him to roll over too. If all the other creditors choose to withdraw, then his expected payoff from withdrawal is

$$\theta(L + ly_0) + (1 - \theta),$$

while his expected payoff from choosing rollover is

$$(1 - \theta) U(y_0)$$

because once the bank is forced into a premature liquidation, the liquidation value is not sufficient to pay off the other outgoing creditors and the creditor who chooses to roll over gets zero. Under a sufficient condition that

$$\frac{\theta}{1 - \theta} L > \frac{r - \rho}{\rho + \phi}$$

it is optimal for the creditor to withdraw with other creditors.

## A.3 Proof of Proposition 3

Suppose that all the other creditors always choose to withdraw in the future. When an individual creditor needs to make his rollover decision, his payoff from withdrawal is 1, and his value function from always choosing rollover, based on the random debt maturity, is $\frac{r+\phi\min(y,1)+\theta\delta(L+ly)}{\rho+\phi+\theta\delta}$. Define

$$y_h^c = \min\{y : r + \phi\min(y, 1) + \theta\delta(L + ly) \geq \rho + \phi + \theta\delta\}.$$

Thus, for $y > y_h^c$, rollover is optimal for the creditor even if the other creditors choose to withdraw. For $y \leq y_h^c$, withdrawal is optimal if other creditors always choose withdrawal in the future.

Now suppose that all the other creditors always choose to roll over in the future. When an individual creditor needs to make his rollover decision, his payoff from withdrawal is 1, and his value function from always choosing rollover, based on the random debt maturity, is $\frac{r+\phi\min(y,1)}{\rho+\phi}$. Define

$$y_l^c = \max\{y : r + \phi\min(y, 1) \geq \rho + \phi\}.$$

Thus, for $y < y_l^c$, withdrawal is optimal for the creditor even if the other creditors choose to roll over; for $y \geq y_l^c$, rollover is optimal if the other creditors always choose to roll over in the future.

Since $l$ is close to 0, and $L < 1$, it is easy to show that $y_l^c < y_h^c$. Therefore when $y \in [y_l^c, y_h^c]$, a creditor finds both rollover and withdrawal are optimal depending on other creditors' strategy.

## A.4 Proof of Proposition 4

**1.** $\mu > 0$ **Case.**

When $\mu > 0$, the bank fundamental will eventually travel to the upper dominance region, in which all creditors will always choose to roll over independent of other creditors' strategy. Let us first consider the value function of a creditor who is locked in by his current contract, by assuming that all creditors in the future will rollover:

$$V^R(y) = E\left[\int_0^{\tau_\phi} e^{-\rho t} r dt + e^{-\rho\tau_\phi} \min\left(y_{\tau_\phi}, 1\right)\Big|\, y_0 = y\right] \tag{13}$$

It is easy to see that $V^R(y)$ is increasing with $y$ and $V^R(1) = \frac{r+\phi}{\rho+\phi} > 1$. Define $y_{\mu+}$ as the solution to the unique equation

$$V^R(y) = 1.$$

It is clear that $y_{\mu+} < 1$. When $y > y_{\mu+}$, it is optimal for a maturing creditor to choose rollover since he can only obtain 1 from withdrawal. It is important to note that the anticipation of the fundamental reaching the upper dominance region prevents another equilibrium in which the creditors all choose withdrawal.

Next, we verify that it is optimal for a maturing creditor to choose withdrawal when $y \leq y_{\mu+} < 1$, if other creditors use a monotone strategy with threshold $y_{\mu+}$. We just need to show that $V^W(y) < 1$ for $y < y_{\mu+}$. Note that in this region, $V^R$ satisfies

$$(\rho + \phi + (1+\theta)\delta) V^R = \mu y V_y^R + r + \phi y + \theta\delta(L + ly) + \delta. \tag{14}$$

Solving this equation provides that $V^R(0) = \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}$. Furthermore, at $y = y_{\mu+}$, $V^R(y_{\mu+}) = 1$. Parameter restrictions (1) and (4) imply that

$$\frac{r + \phi + \theta\delta(L + l) + \delta}{\rho + \phi + (1+\theta)\delta} < 1,$$

which in turn provides that $V^R(0) < 1$. Therefore, if $V^R(y) > 1$ for some $y < y_{\mu+}$, then we must have some point $\widehat{y}$ such that $V^R(\widehat{y}) > 1$ and $V_y^R(\widehat{y}) = 0$. But then equation (14) implies that

$$V^R\left(\widehat{y}\right) < \frac{r+\phi+\theta\delta\left(L+l\right)+\delta}{\rho+\phi+\left(1+\theta\right)\delta} < 1,$$

which contradicts with $V^R\left(\widehat{y}\right) > 1$. Thus, it is optimal for an individual creditor to withdraw when $y \le y_{\mu+}$.

This monotone equilibrium is unique, because there is only one $y_{\mu+}$ to satisfy $V^R\left(y_{\mu+}\right) = 1$ in Eq. (13). Clearly $y_{\mu+}$ does not depend on $L$.

**2. $\mu < 0$ Case.**

When $\mu < 0$, the bank fundamental will eventually travel to the lower dominance region, in which all creditors will always choose to withdraw independent of other creditors' strategy. We first consider the value function $V^W\left(y\right)$ of a creditor who is locked in by his current contract, by assuming that all the other creditors will withdraw in the future. $V^W$ satisfies

$$\left(\rho+\phi+\left(1+\theta\right)\delta\right)V^W = \mu y V_y^W + r + \phi \min\left(1,y\right) + \theta\delta\left(L+ly\right) + \delta, \tag{15}$$

with the boundary condition $V^W\left(0\right) = \frac{r+\theta\delta L+\delta}{\rho+\phi+\left(1+\theta\right)\delta} < 1$. It is easy to show that $V^W$ is increasing with $y$, therefore there exists $y_{\mu-}$ such that

$$V^W\left(y_{\mu-}\right) = 1.$$

For $y < 1$, the general solution to Eq. (15) is

$$V^W\left(y\right) = \frac{r+\theta\delta L+\delta}{\rho+\phi+\left(1+\theta\right)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+\left(1+\theta\right)\delta-\mu}y + Ay^{\frac{\rho+\phi+\left(1+\theta\right)\delta}{\mu}}$$

where $A$ is constant. Because $\mu < 0$, $A$ has to be zero otherwise $V^W\left(0\right)$ diverges. Therefore $V^W\left(1\right) < \frac{r+\phi+\theta\delta\left(L+l\right)+\delta}{\rho+\phi+\left(1+\theta\right)\delta} < 1$, which in turn implies that $y_{\mu-} > 1$. Since an individual creditor gets 1 from choosing withdrawal, it is optimal to withdraw if $y < y_{\mu-}$. It is again important to note that the anticipation of the fundamental reaching the lower dominance region prevents another equilibrium in which the creditors all choose rollover.

Next, we need to verify that if $y > y_{\mu-}$, rollover is optimal if the other creditors use a rollover threshold $y_{\mu-}$. In this region, the creditor's value function satisfies

$$\left(\rho+\phi\right)V^W = \mu y V_y^W + r + \phi,$$

with the boundary condition $V^W\left(y_{\mu-}\right) = 1$. The solution is $\frac{r+\phi}{\rho+\phi} + By^{\frac{\rho+\phi}{\mu}}$ where $B < 0$ is constant. Note that this function is monotonically increasing. Thus, $V^W\left(y\right) > 1$ if $y > y_{\mu-}$. In other words, rollover is optimal.

### A.5 Proof of Theorem 5

Suppose that the single creditor uses a rollover threshold $y_s$. Then his value function $V^s(y; y_s)$, which takes the threshold $y_s$ as given, satisfies the following differential equation:

$$\begin{cases} [\rho+\phi] V^s(y) = \frac{\sigma^2}{2} y^2 V^s_{yy} + \mu y V^s_y + \phi \min(1, y) + r & \text{if } y \geq y_s \\ [\rho+\phi+\delta] V^s(y) = \frac{\sigma^2}{2} y^2 V^s_{yy} + \mu y V^s_y + \phi \min(1, y) + r + \delta(L + ly) & \text{if } y < y_s \end{cases}$$

Here we assume that $L + l y_s < 1$, which we will verify later.

The characteristic equation of the differential equation in the region $y \geq y_s$ is given in equation (10) whose two roots are $-\gamma_2$ and $\eta_2$. The characteristic equation of the differential equation in the region $y < y_s$ is

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - [\rho+\phi+\delta] = 0,$$

which has two solutions:

$$-\gamma_3 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2[\rho+\phi+\delta]}}{\sigma^2} < 0$$

and

$$\eta_3 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2[\rho+\phi+\delta]}}{\sigma^2} > 1.$$

We can solve the Bellman equation following the same procedure as in the proof of Lemma 9. Depending on whether $y_s$ is above or below 1, we have two cases.

When $y_s < 1$.

$$V^s(y; y_s) = \begin{cases} \frac{r+\delta L}{\rho+\phi+\delta} + \frac{\phi+\delta l}{\rho+\phi+\delta-\mu} y + B_1 y^{\eta_3} & \text{when } 0 < y < y_s \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y + A_2 y^{-\gamma_2} + B_2 y^{\eta_2} & \text{when } y_s < y < 1 \\ \frac{r+\phi}{\rho+\phi} + A_3 y^{-\gamma_2} & \text{when } 1 < y \end{cases}$$

The four coefficients $B_1$, $B_2$, $A_2$, and $A_3$ are given by

$$\begin{aligned} B_1 &= \frac{[K_1\gamma_2 + H_1] - y_s^{-\eta_2}(\gamma_2 K_2 + H_2 y_s)}{(\eta_3+\gamma_2) y_s^{\eta_3-\eta_2}} \\ B_2 &= \frac{y_s^{-\eta_2}}{\eta_2+\gamma_2}\left[\gamma_2 K_2 + H_2 y_s + B_1(\eta_3+\gamma_2) y_s^{\eta_3}\right] \\ &= \frac{1}{\eta_2+\gamma_2}[K_1\gamma_2 + H_1] \\ A_2 &= \frac{y_s^{\gamma_2}}{\eta_2+\gamma_2}\left[\eta_2 K_2 - H_2 y_s + B_1(\eta_2-\eta_3) y_s^{\eta_3}\right] \\ A_3 &= A_2 - \frac{1}{\eta_2+\gamma_2}[K_1\eta_2 - H_1] \end{aligned}$$

where

$$
\begin{aligned}
K_2 &= \frac{r+\delta L}{\rho+\phi+\delta} - \frac{r}{\rho+\phi} + H_2 y_s \\
K_1 &= -\frac{\phi\mu}{(\rho+\phi)(\rho+\phi-\mu)} \\
H_1 &= -\frac{\phi}{\rho+\phi-\mu} \\
H_2 &= \frac{\delta l(\rho+\phi-\mu)-\phi\delta}{(\rho+\phi+\delta-\mu)(\rho+\phi-\mu)} < 0
\end{aligned}
$$

When $y_s > 1$

$$
V^s(y;y_s) = \begin{cases} \frac{r+\delta L}{\rho+\phi+\delta} + \frac{\phi+\delta l}{\rho+\phi+\delta-\mu} y + D_1 y^{\eta_3} & \text{when } 0 < y < 1 \\ \frac{r+\phi+\delta L}{\rho+\phi+\delta} + \frac{\delta l}{\rho+\phi+\delta-\mu} y + C_2 y^{-\gamma_3} + D_2 y^{\eta_3} & \text{when } 1 < y < y_s \\ \frac{r+\phi}{\rho+\phi} + C_3 y^{-\gamma_2} & \text{when } y_s < y \end{cases}
$$

The four coefficients $D_1$, $D_2$, $C_2$, and $C_3$ are given by

$$
\begin{aligned}
C_2 &= \frac{M_2\eta_3 - M_1}{(\eta_3+\gamma_3)} \\
C_3 &= \frac{\eta_3+\gamma_3}{\eta_3+\gamma_2} C_2 y_s^{\gamma_2-\gamma_3} + \frac{\eta_3-1}{\eta_3+\gamma_2} \frac{\delta l}{\rho+\phi+\delta-\mu} y_s^{\gamma_2+1} \\
&\quad -\frac{\eta_3}{\eta_3+\gamma_2}\left[\frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\delta L}{\rho+\phi+\delta}\right] y_s^{\gamma_2} \\
&= \frac{(\eta_3+\gamma_3) C_2 (y_s)^{-\gamma_3} - \eta_3 N_1 - \frac{\delta l}{\rho+\phi+\delta-\mu} y_s}{(\eta_3+\gamma_2) y_s^{-\gamma_2}} \\
D_2 &= \frac{(\gamma_3-\gamma_2) C_2 (y_s)^{-\gamma_3} + \gamma_2 N_1 - \frac{\delta l}{\rho+\phi+\delta-\mu} y_s}{(\eta_3+\gamma_2) y_s^{\eta_3}} \\
D_1 &= D_2 - \frac{M_2\gamma_3 + M_1}{(\eta_3+\gamma_3)}
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= \frac{\phi}{\rho+\phi+\delta-\mu} \\
M_2 &= \frac{\phi\mu}{(\rho+\phi+\delta)(\rho+\phi+\delta-\mu)} \\
N_1 &= \frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\delta L}{\rho+\phi+\delta} - \frac{\delta l}{\rho+\phi+\delta-\mu} y_s
\end{aligned}
$$

Next, we show that there exists a unique threshold $y_s$ such that $V^s(y_s) = \min(L + l y_s, 1)$.

Define $W^s(y_s) = V^s(y_s; y_s)$. When $y_s < 1$

$$\begin{aligned}
W^s(y_s) &= \frac{r+\delta L}{\rho+\phi+\delta} + \frac{\phi+\delta l}{\rho+\phi+\delta-\mu} y_s + \frac{[K_1\gamma_2 + H_1] - y_s^{-\eta_2}(\gamma_2 K_2 + H_2 y_s)}{(\eta_3+\gamma_2) y_s^{-\eta_2}} \\
&= \frac{r+\delta L}{\rho+\phi+\delta} + \frac{\phi+\delta l}{\rho+\phi+\delta-\mu} y_s + \frac{[K_1\gamma_2 + H_1] y_s^{\eta_2} - (\gamma_2 K_2 + H_2 y_s)}{(\eta_3+\gamma_2)} \\
&= \frac{\eta_3}{\eta_3+\gamma_2}\frac{r+\delta L}{\rho+\phi+\delta} - \frac{\gamma_2}{\eta_3+\gamma_2}\frac{r}{\rho+\phi} + \\
&\quad \left[\frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\delta l+\phi}{\rho+\phi+\delta-\mu} + \frac{1+\gamma_2}{(\eta_3+\gamma_2)}\frac{\phi}{(\rho+\phi-\mu)}\right] y_s + \frac{[K_1\gamma_2 + H_1] y_s^{\eta_2}}{\eta_3+\gamma_2}
\end{aligned}$$

Following the same steps as in the proof of Proposition 10, we can show that $dW^s(y_s)/dy^s - l > 0$ when $l$ is sufficiently small.

When $y_s > 1$

$$\begin{aligned}
W(y_s) &= \frac{r+\phi}{\rho+\phi} + C_3 y^{-\gamma_2} \\
&= \frac{\eta_3+\gamma_3}{\eta_3+\gamma_2} C_2 y^{-\gamma_3} + \frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\delta l}{\rho+\phi+\delta-\mu} y \\
&\quad + \frac{\gamma_2}{\eta_3+\gamma_2}\frac{r+\phi}{\rho+\phi} + \frac{\eta_3}{\eta_3+\gamma_2}\frac{r+\phi+\delta L}{\rho+\phi+\delta}
\end{aligned}$$

We want to put parameter restrictions that when $y_s = \frac{1-L}{l}$, we have

$$\begin{aligned}
& W\left(\frac{1-L}{l}\right) - 1 \\
=\ & \frac{\eta_3+\gamma_3}{\eta_3+\gamma_2} C_2 \left(\frac{1-L}{l}\right)^{-\gamma_3} + \frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\delta(1-L)}{\rho+\phi+\delta-\mu} + \frac{\gamma_2}{\eta_3+\gamma_2}\frac{r+\phi}{\rho+\phi} + \frac{\eta_3}{\eta_3+\gamma_2}\frac{r+\phi+\delta L}{\rho+\phi+\delta} - 1 \\
=\ & \frac{M_2\eta_3 - M_1}{\eta_3+\gamma_2}\left(\frac{1-L}{l}\right)^{-\gamma_3} + \frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\mu\delta(1-L)}{(\rho+\phi+\delta-\mu)(\rho+\phi+\delta)} + \frac{\gamma_2}{\eta_3+\gamma_2}\frac{r+\phi}{\rho+\phi} \\
& + \frac{\eta_3}{\eta_3+\gamma_2}\frac{r+\phi+\delta}{\rho+\phi+\delta} - \frac{1}{\eta_3+\gamma_2}\frac{\delta(1-L)}{\rho+\phi+\delta} - 1 \\
>\ & \frac{M_2\eta_3 - M_1}{\eta_3+\gamma_2}\left(\frac{1-L}{l}\right)^{-\gamma_3} + \frac{1}{\rho+\phi+\delta}\left(r+\phi+\delta - \frac{\delta(1-L)}{\eta_3+\gamma_2}\right) - 1
\end{aligned}$$

Note that $\frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\delta(1-L)}{\rho+\phi+\delta-\mu} = \frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\mu\delta(1-L)}{(\rho+\phi+\delta-\mu)(\rho+\phi+\delta)} + \frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\delta(1-L)}{\rho+\phi+\delta}$, therefore

$$\begin{aligned}
W\left(\frac{1-L}{l}\right) - 1 &= \frac{M_2\eta_3 - M_1}{\eta_3+\gamma_2}\left(\frac{1-L}{l}\right)^{-\gamma_3} + \frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\mu\delta(1-L)}{(\rho+\phi+\delta-\mu)(\rho+\phi+\delta)} \\
&\quad + \frac{\gamma_2}{\eta_3+\gamma_2}\frac{r+\phi}{\rho+\phi} + \frac{\eta_3}{\eta_3+\gamma_2}\frac{r+\phi+\delta}{\rho+\phi+\delta} - \frac{1}{\eta_3+\gamma_2}\frac{\delta(1-L)}{\rho+\phi+\delta} - 1 \\
&> \frac{M_2\eta_3 - M_1}{\eta_3+\gamma_2}\left(\frac{1-L}{l}\right)^{-\gamma_3} + \frac{\eta_3-1}{\eta_3+\gamma_2}\frac{\mu\delta(1-L)}{(\rho+\phi+\delta-\mu)(\rho+\phi+\delta)} \\
&\quad + \frac{1}{\rho+\phi+\delta}\left(r+\phi+\delta - \frac{\delta(1-L)}{\eta_3+\gamma_2}\right) - 1
\end{aligned}$$

When $l \to 0$, the first term is negligible, and to ensure $W\left(\frac{1-L}{l}\right) > 1$ we need

$$r + \frac{\eta_3 - 1}{\eta_3 + \gamma_2} \frac{\mu\delta(1-L)}{(\rho + \phi + \delta - \mu)} > \frac{\delta(1-L)}{\eta_3 + \gamma_2} + \rho$$

which requires that the project is sufficiently profitable.

With this result we can show that $W(y_s)$ has at most one intersection with $\min(L + ly_s, 1)$. Note that when $y_s > 1$, $W(y_s)$ is concave and increasing; therefore it cannot be the case that the solution occurs for $y_s > \frac{1-L}{l}$. We have two cases to discuss. First, suppose that there exists $y_s < 1$ such that $W(y_s) = L + ly_s$. Then $W(1) > L + ly_s$. Since $W\left(\frac{1-L}{l}\right) > 1$, the concavity of $W(y_s)$ implies that $W(y)$ is always above the line $L + ly$ for $y_s \in \left[1, \frac{1-L}{l}\right]$. But for $y_s > \frac{1-L}{l}$, $W(y_s) > 1$ because of it is increasing. Second, suppose that $W(1) < L + l$. Then for $y_s \in \left[1, \frac{1-L}{l}\right]$ there exists exactly one solution for $W\left(\frac{1-L}{l}\right) = L + ly_s$ because $W\left(\frac{1-L}{l}\right) - L - ly_s$ is concave and have opposite signs on each end point.

Finally we show the optimality of threshold strategy. We omit the policy dependence by writing $V^s(y; y_s) = V^s(y)$.

For $y_s < 1$ we first show that

$$V_y^{s\prime}(y_s) > l$$

Otherwise, because $V^s(\infty)$ approaches $\frac{r+\phi}{\rho+\phi} > 1$, there must exist two points $y_1 < y_s < y_2$ such that

$$V_y^s(y_1) = V_y^s(y_2) = l, \text{ and } V_{yy}^s(y_1) < 0 < V_{yy}^s(y_2)$$

and

$$V^s(y_1) > L + ly_1 \text{ and } V^s(y_2) < L + ly_2,$$

Since $V^s(y)$ is concave for $y > 1$, $y_2 < 1$. $V^s(y_1) > L + ly_1$ implies that

$$\begin{aligned} &\frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}^s(y_1) + \mu y_1 l + r + \phi y_1 + \delta L + \delta l y_1}{\rho + \phi + \delta} > L + ly_1 \\ \Rightarrow \quad & r + \phi y_1 > L(\rho + \phi) + (\rho + \phi - \mu) l y_1 \end{aligned}$$

Similarly $V^s(y_2) < L + ly_2$ implies that

$$\begin{aligned} &\frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}^s(y_2) + \mu y_2 l + r + \phi y_2}{\rho + \phi} < L + ly_2 \\ \Rightarrow \quad & r + \phi y_2 < L(\rho + \phi) + (\rho + \phi - \mu) l y_1 \end{aligned}$$

As $l$ can be arbitrarily small, we have a contradiction. Given this result, applying similar methods as in proof of Lemma 10 shows that $V^s(y)$ is increasing for $y > y^s$. For $y < y^s$, notice that $V^s(y)$ is either concave or convex, depending on whether $B_1 < 0$ or not. But both cases implies that $V^s(y) < V^s(y^s)$.

For $y_s > 1$. For sufficiently small $l$, $V^s\left(\frac{1-L}{y}\right) > 1$. Then since $V^s$ in concave for $y > y_s$ and $V^s(\infty) > 1$, we have $V_y^{s\prime}(y_s) > l$ and any non-monotonic behavior only can occur in the region of $[0, y_s]$. Say there exists $y_1 < y_s < y_2$ such that

$$V_y^s(y_1) = V_y^s(y_2) = l, \text{ and } V_{yy}^s(y_1) < 0 < V_{yy}^s(y_2)$$

and

$$V^s(y_1) > L + ly_1 \text{ and } V^s(y_2) < L + ly_2.$$

Then we must have

$$\begin{aligned} r + \phi \min(y_1, 1) &> L(\rho + \phi) + (\rho + \phi - \mu) l y_1; \\ r + \phi \min(y_2, 1) &< L(\rho + \phi) + (\rho + \phi - \mu) l y_2. \end{aligned}$$

When $y_1 < y_2 < 1$, contradiction is immediate as $l$ is sufficiently small. When $y_2 > 1$, since $ly_2 < 1 - L$, to obtain contradiction for the second equation we only need that $r > \rho + \mu(1 - L)$.

## A.6 Proof of Proposition 6

We first show the multiple creditor case. Note that $y_*$ is determined by the condition that $V(y_*; y_*) = 1$. Theorem 1 implies that if $y_* > 1$, it is determined by the following implicit function:

$$\begin{aligned} G(y_*) &= \frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} C_2 y_*^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1+\theta)\delta - \mu} y_* \\ &\quad + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1+\theta)\delta} \\ &= 1 \end{aligned}$$

By the implicit function theorem, we have

$$\begin{aligned} \frac{dy_*}{dL} &= -\frac{\partial G / \partial L}{\partial G / \partial y_*} \\ &= -\frac{\eta_1 \frac{\theta \delta}{\rho + \phi + (1+\theta)\delta}}{-\gamma_1 (\eta_1 + \gamma_1) C_2 y_*^{-\gamma_1 - 1} + (\eta_1 - 1) \frac{\theta \delta l}{\rho + \phi + (1+\theta)\delta - \mu}} < 0. \end{aligned} \tag{16}$$

If $y_* < 1$, it is determined by

$$\begin{aligned} F(y_*) &= \frac{\eta_1}{(\eta_1 + \gamma_2)} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1+\theta)\delta} - \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r}{\rho + \phi} + \frac{[K_1 \gamma_2 + H_1]}{(\eta_1 + \gamma_2)} y_*^{\eta_2} \\ &\quad + \left[ \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l + \phi}{\rho + \phi + (1+\theta)\delta - \mu} + \frac{1 + \gamma_2}{(\eta_1 + \gamma_2)} \frac{\phi}{(\rho + \phi - \mu)} \right] y_* = 1 \end{aligned}$$

By the implicit function theorem, we have

$$\begin{aligned}\frac{dy_*}{dL} &= -\frac{\partial F/\partial L}{\partial F/\partial y_*} \\ &= -\frac{\frac{\eta_1\theta\delta}{\rho+\phi+(1+\theta)\delta}}{\frac{\eta_1-1}{\eta_1+\gamma_2}\frac{\theta\delta l+\phi}{\rho+\phi+(1+\theta)\delta-\mu}+\frac{1+\gamma_2}{(\eta_1+\gamma_2)}\frac{\phi}{(\rho+\phi-\mu)}+\eta_1\left(K_1\gamma_2+H_1\right)y_*^{\eta_2-1}} < 0. \qquad (17)\end{aligned}$$

Taken together, the equilibrium rollover threshold $y_*$ decreases with $L$.

The single creditor case is similar, but with a different implicit function $F(y_s) = W^s(y_s) - L - ly_s$. Therefore $\frac{dy_s}{dL} = -\frac{\partial F/\partial L}{\partial F/\partial y_s} = -\frac{dW^s(y_s)/dL-1}{\partial F/\partial y_s}$. It amounts to showing that the numerator is negative, and indeed both $y_s < 1$ and $y_s > 1$ cases have

$$dW^s(y_s)/dL - 1 = \frac{\eta_3}{\eta_3+\gamma_2}\frac{\delta}{\rho+\phi+\delta} - 1 < 0.$$

## A.7 Proof of Proposition 7

When $y_*$ is large ($y_* >> 1$), equation (16) directly implies that

$$\frac{dy_*}{dL} \simeq -\frac{\eta_1}{(\eta_1-1)}\frac{\rho+\phi+(1+\theta)\delta-\mu}{\rho+\phi+(1+\theta)\delta}\frac{1}{l}.$$

When $y_*$ is small ($y_* << 1$), equation (16) directly implies that

$$\frac{dy_*}{dL} \simeq -\frac{\frac{\eta_1\theta\delta}{\rho+\phi+(1+\theta)\delta}}{\frac{(\eta_1-1)(\theta\delta l+\phi)}{\rho+\phi+(1+\theta)\delta-\mu}+\frac{(1+\gamma_2)\phi}{(\rho+\phi-\mu)}} \quad \text{when } y_* \; << 1$$

## A.8 Proof of Proposition 8

We distinguish between an individual creditor $i$'s rollover frequency $\delta_i$ and other creditors' rollover frequency $\delta_{-i}$. We can rewrite the individual creditor's Bellman equation for his value function $V^i$:

$$\begin{aligned}\rho V^i(y_t;y_*) &= \mu y_t V_y^i + \frac{\sigma^2}{2}y_t^2 V_{yy}^i + r + \phi\left[\min(1,y_t) - V(y_t;y_*)\right] \qquad (18)\\ &\quad +\theta\delta_{-i}\mathbf{1}_{\{y<y_*\}}\left[\min(L+ly_t,1) - V(y_t;y_*)\right] + \delta_i \max_{\text{rollover or run}}\left\{1 - V(y_t;y_*),0\right\}.\end{aligned}$$

Suppose that we increase $\delta_i$ from $\delta$ to $\delta' > \delta$. We need to show that the creditor $i$'s value function with parameter $\delta'$ to that with parameter $\delta$. To facilitate the comparison, we consider a new problem, in which while the creditor's contract expires with rate $\delta'$, he is only allowed to withdraw at his contract expiration if an independent random variable $X = 1$. This variable $X$ can take values of 1 or 0 with probabilities of $\lambda = \delta/\delta' < 1$ and $1-\lambda$, respectively. This random variable effectively reduces the creditor's release rate to $\delta$. Thus, in this constrained problem with parameter $\delta'$, the creditor has the same value function as in the unconstrained problem with parameter $\delta$.

Next, consider the creditor's value function in the unconstrained problem with paramter $\delta'$, which should be strictly higher than that in the constrained problem. This is because if the creditor is allowed to withdraw when $X = 0$ and $y_t < y_*$, his value function is strictly increased even if he keeps the same threshold. Then, it is obvious that the creditor's value function in the unconstrained problem with parameter $\delta'$ is strictly higher than that in the same problem with parameter $\delta$.

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